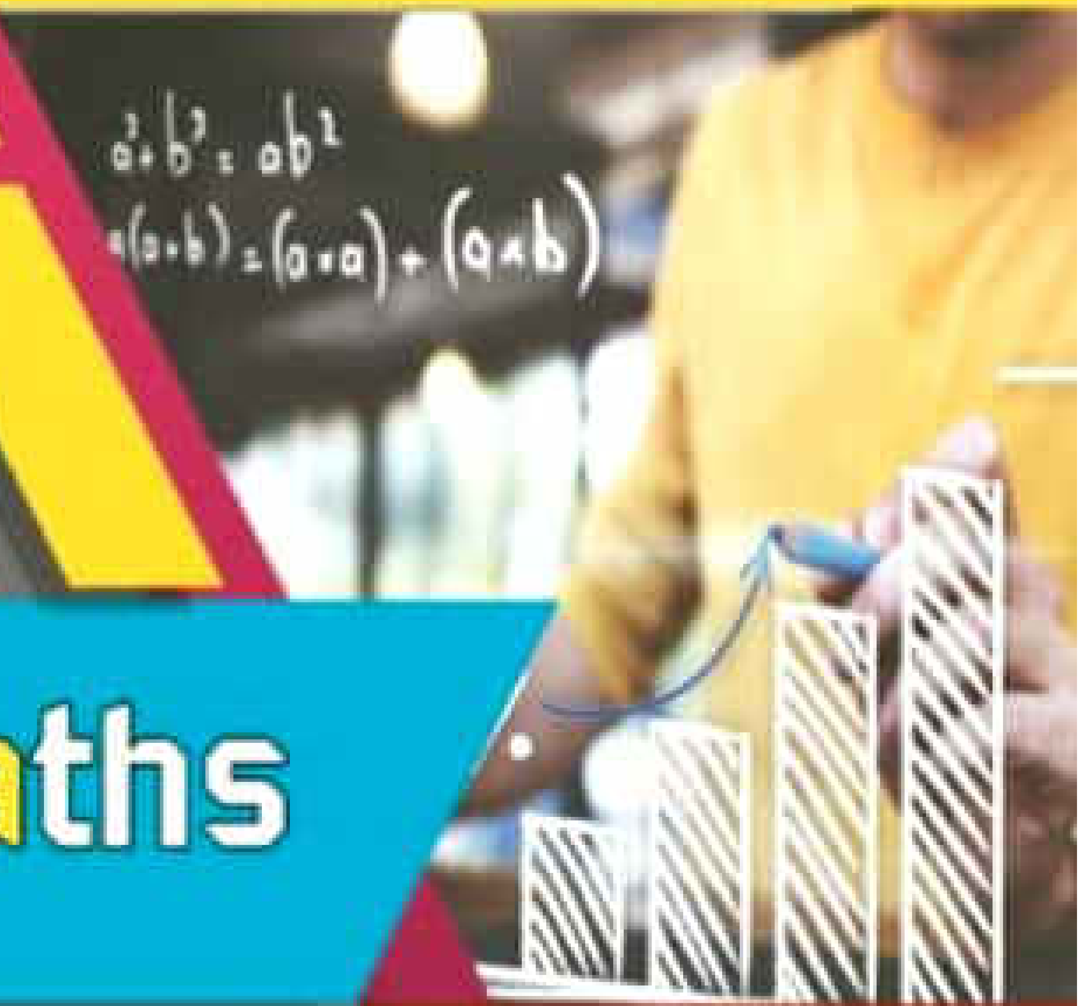
KANZY



In Maths

By Mr

Ahmed Omar

معلم أول رياضيات

01023636682

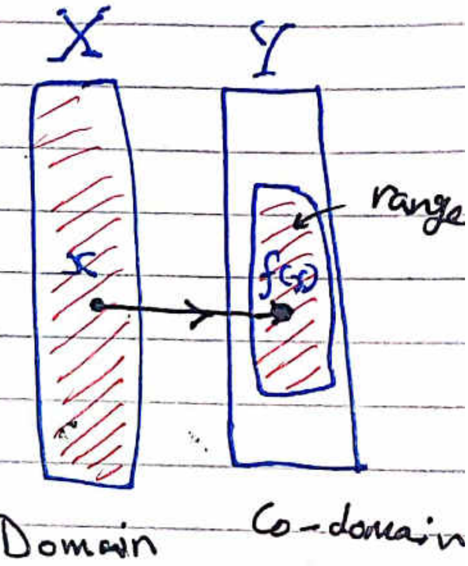
Algebra:

lesson 1, Real functions

Definition of the function:

Let X, Y be two non-empty sets.

The relation from the set X to the set Y is a function if each element in X is related with one and only one element in Y



* The range: the set of elements of the co-domain which each of them has an origin in the domain

note that: the range $\subset$ the co-domain

* The Real function:

The function $f: X \to Y$ is called a real function if each of the domain (X) and the co-domain (Y) is the set of real numbers or a proper subset of $\mathbb{R}$

Example:

Determine which of The following relations represents a rule of function in x:

1) $y = 2x + 5$

Solution: Let $x = 1 \Rightarrow y = 7, \ldots$

The relation is function because every real value of the variable x corresponds a unique value of the variable y

2) $y^2 = x + 4$

Solution: Let $x = 1 \Rightarrow y^2 = 5 \Rightarrow y = \pm\sqrt{5}$

is not a function because there is at least one real value of the variable x corresponds two different values of the variable y

3) $y = \sqrt{x^2 + 4}$

Solution: let $x = 1 \Rightarrow y = \sqrt{5}, \ldots$

the relation is a function

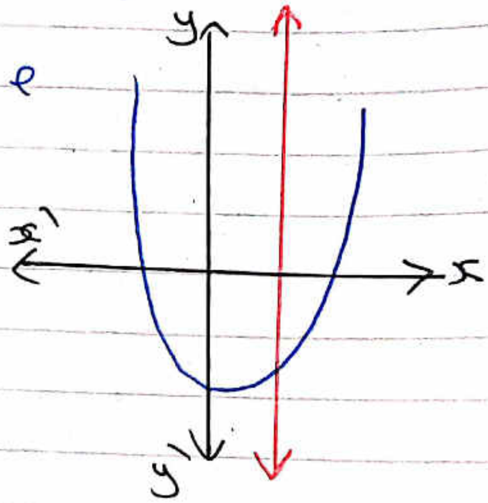
3

Test of the vertical line:

1 Function

The relation represents a function

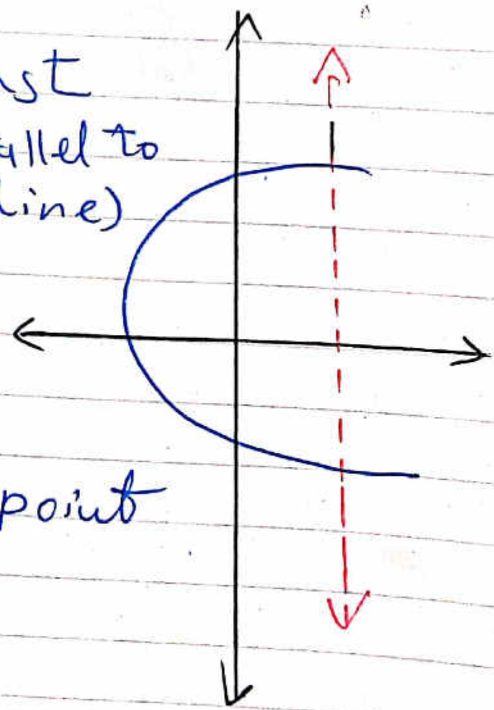
for each vertical line intersects the curve at one point at most



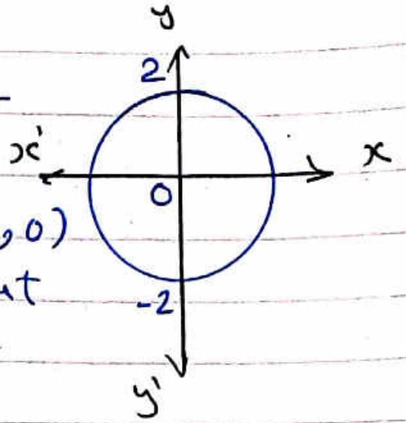
2 not Function

The relation is not a function

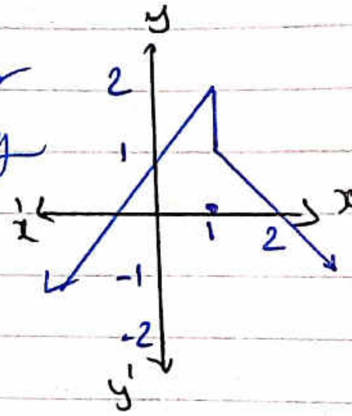
If there exists at least one straight line parallel to y-axis (vertical line) and intersects the graph of the relation at more than one point



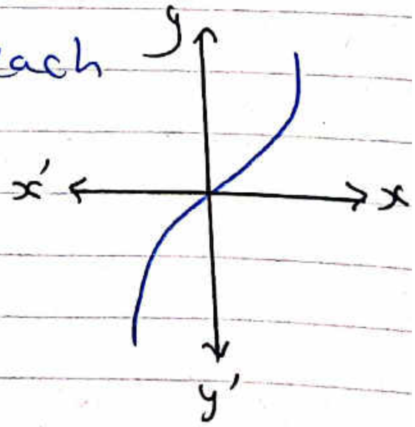
Example: In each of the following graphs, show if y is a function in x or not:



1) Doesn't represent a function for there is a vertical line passing through the point $(0,0)$ and intersect the curve at $(0,2)$, $(0,-2)$

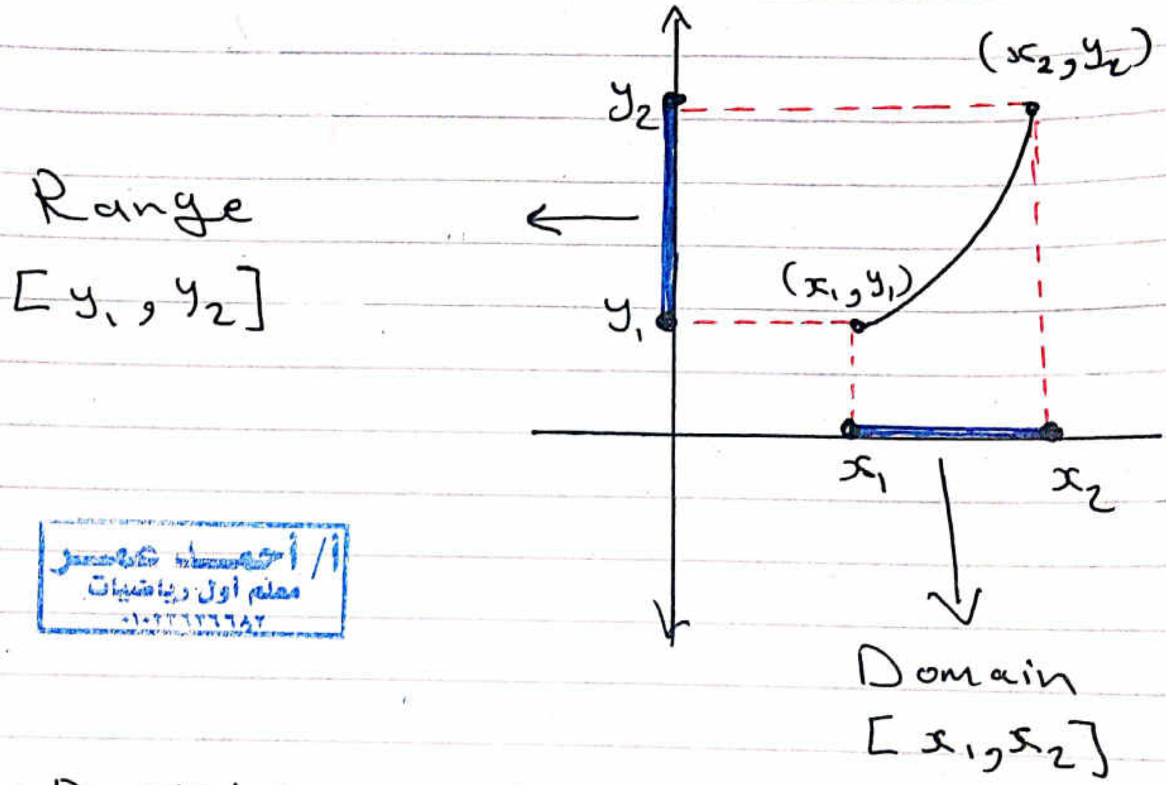


2) Doesn't represent a function for there is a vertical line passing through the point $(1,0)$ and intersects the curve at a set of points.



3) Represents a function for each vertical line intersects the curve at one point at most

* Determining The domain and the range from a graph:



The Domain:

is The set of the x- Coordinates of all the points that lie on the curve of the function.

the Range:

is The set of The y-Coordinates of all points that lie on the curve of The function

Graph the function f, then from the graph deduce its range where:

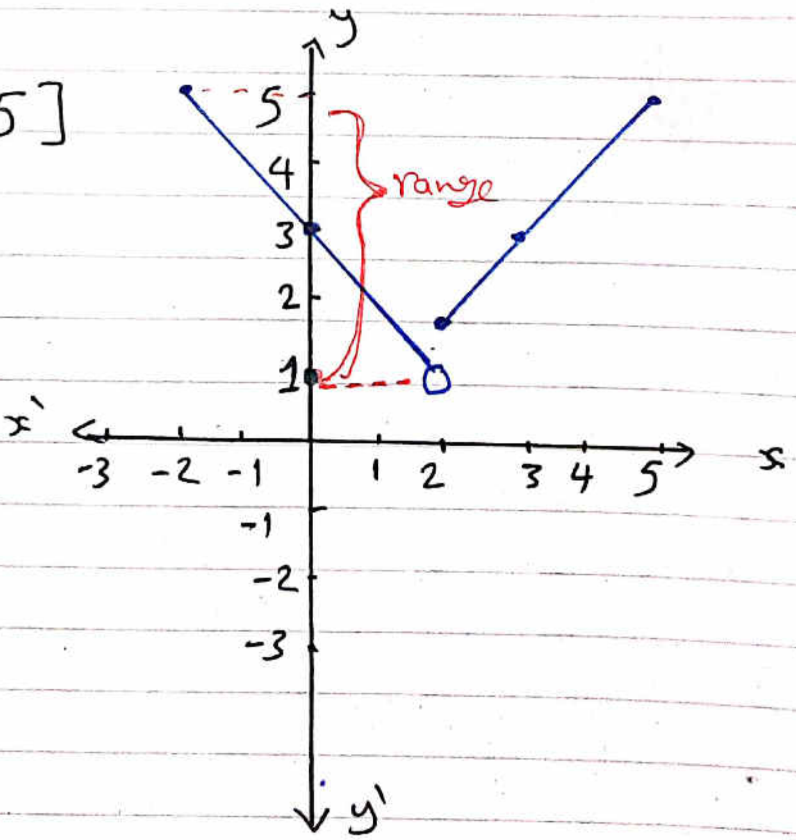
(1) $f(x) = \begin{cases} 3 - x & , -2 \le x < 2 \\ x & , 2 \le x \le 5 \end{cases}$

Solution:

$f_1(x) = 3 - x$ | $f_2(x) = x$

-2	0	2	2	3	5
5	3	1	2	3	5

The Range =]1, 5]



(2) $f(x) = \begin{cases} x - 1, & 2 < x \leq 4 \\ -1, & -2 \leq x \leq 2 \end{cases}$

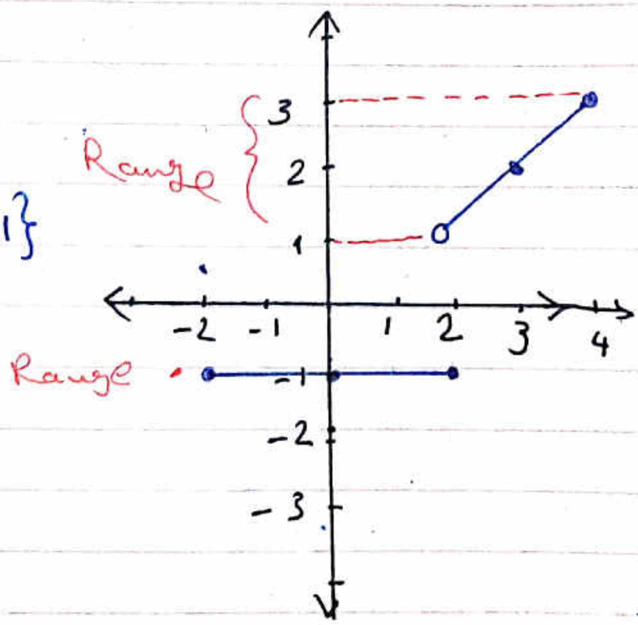
Solution:

$f_1(x) = x - 1$ | $f_2(x) = -1$

(2)	3	4	-2	0	2
(1)	2	3	-1	-1	-1

The range

$=]1, 3] \cup \{-1\}$



3

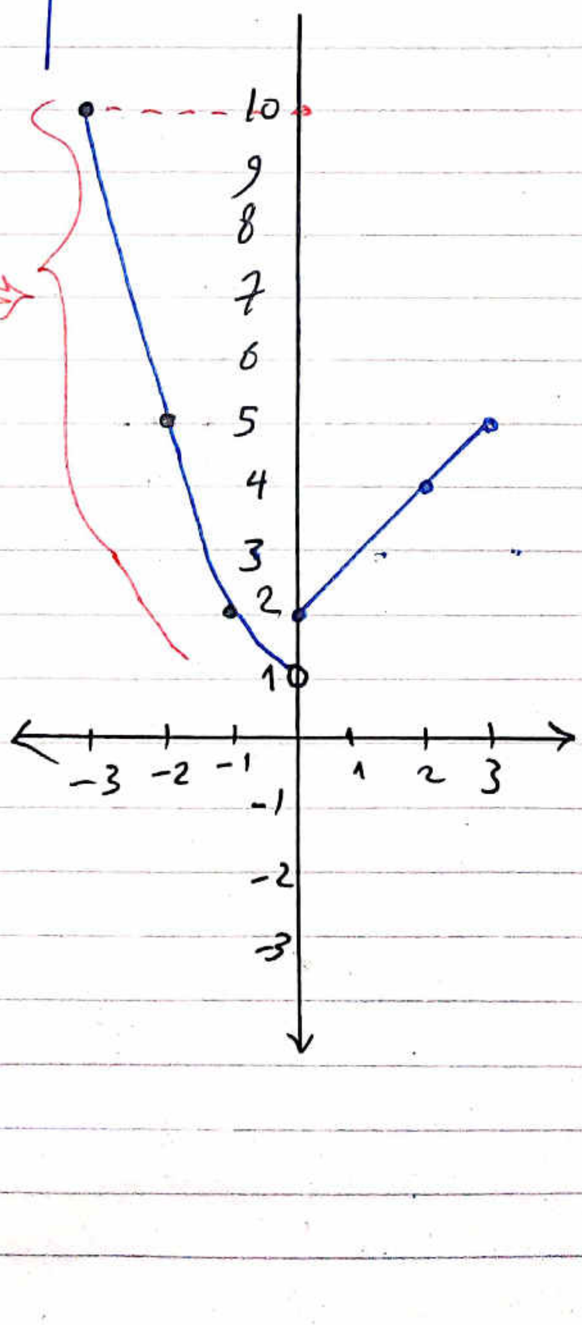
$$f(x) = \begin{cases} x^2 + 1 & , -3 \le x < 0 \\ x + 2 & , 0 \le x \le 3 \end{cases}$$

Solution

$f_1(x) = x^2 + 1$ | $f_2(x) = x + 2$

-3	-2	-1	0	0	2	3
10	5	2	1	2	4	5

The range $=]1, 10]$



Identifying the domain of the real functions:

1 polynomial function:

The domain of the polynomial function is $\mathbb{R}$ unless it is defined on a subset of it

2 Rational function:

If $f(x) = \frac{h(x)}{g(x)}$

then the domain of the function f

$= \mathbb{R}$ − the set of zeroes of the denominator.

3 If $f(x) = \sqrt[n]{h(x)}$ where $n \in \mathbb{Z}^+$,

$n > 1$, $h(x)$ is a polynomial.

n is called the index of the root

First: when (n) is an odd number, then the domain of $f = \mathbb{R}$

Second: when (n) is an even number then the domain is the set of all values of x which satisfy $h(x) \geq 0$.

Example: Determine the domain of each of the real functions defined by the following rules:

① $f(x) = \dfrac{x+3}{x^2-9}$

Solution: Let $x^2 - 9 = 0$

$\Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

The domain $= \mathbb{R} - \{-3, 3\}$

[2] $f(x) = \dfrac{2x+1}{x^2-3x+2}$

Let $x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0 \Rightarrow \boxed{x=1}$ or $\boxed{x=2}$

the domain $= \mathbb{R} - \{1, 2\}$

[3] $f(x) = \dfrac{x+1}{x^4+x}$

Let: $x^4 + x = 0 \Rightarrow x(x^3+1) = 0$

$\boxed{x=0}$, $x^3+1=0 \Rightarrow x^3 = -1 \Rightarrow \boxed{x=-1}$

The domain $= \mathbb{R} - \{0, -1\}$

④ $f(x) = \dfrac{2x+5}{x^2+x+1}$

$\because x^2 + x + 1 = 0$ has no solution in $\mathbb{R}$

$\therefore$ the domain $= \mathbb{R}$

5) $f(x) = \sqrt{x-3}$

Solution

$\because$ the index of the root is even

$\therefore$ $x - 3 \geq 0 \Rightarrow x \geq 3$

$\therefore$ the domain $= [3, \infty[$

6) $f(x) = \sqrt[3]{x-5}$

$\because$ the index of the root is an odd number

$\therefore$ The domain $= \mathbb{R}$

7) $f(x) = \sqrt[4]{x^2+4}$

$\because$ $x^2+4 \geq 0$ for all values of x

$\therefore$ the domain $= \mathbb{R}$

8) $f(x) = \dfrac{3}{\sqrt{x-3}}$

Solution $\because$ $x - 3 > 0 \Rightarrow x > 3$

The domain $=]3, \infty[$

9) $f(x) = \sqrt{x^2-16}$

$x^2 - 16 \geq 0 \Rightarrow (x-4)(x+4) \geq 0$

The domain $= \mathbb{R} -]-4, 4[$

10) $f(x) = \sqrt{4 - x^2}$

Solution ; $4 - x^2 \geq 0$

$\Rightarrow x^2 - 4 \leq 0 \Rightarrow (x-4)(x+4) \leq 0$

the domain $= [-4, 4]$

11) $f(x) = \dfrac{5}{\sqrt{9 - x^2}}$

Solution ;

$9 - x^2 > 0 \Rightarrow x^2 - 9 < 0$

$\Rightarrow (x-3)(x+3) < 0$

the domain $=]-3, 3[$

12) $f(x) = \dfrac{1}{\sqrt{x^2 - 4x + 4}}$

Solution ;

$x^2 - 4x + 4 > 0$

$\Rightarrow (x-2)^2 > 0$

$\therefore$ the domain $= \mathbb{R} - \{2\}$

13) $f(x) = \dfrac{5}{\sqrt{x} - 1}$

solution $x \geq 0$, $x \neq 1$

the domain $= [0, \infty[- \{1\}$

Lesson 2:
- operations on functions
- Composition of functions.

First: operations on functions:

If f_1, f_2 are two functions whose domain are D_1, D_2 respectively, then:

(1) $(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$

and the domain of $(f_1 \pm f_2)$ is $D_1 \cap D_2$

(2) $(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$,

the domain is $D_1 \cap D_2$

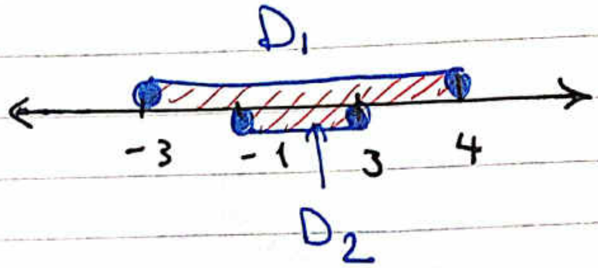
$\boxed{3}$ $\left(\dfrac{f_1}{f_2}\right)(x) = \dfrac{f_1(x)}{f_2(x)}$ such that $f_2(x) \neq$ zero

the domain of $\left(\dfrac{f_1}{f_2}\right)$ is

$(D_1 \cap D_2) - Z(f_2)$

Example: If $f_1(x) = x+2$ and the domain of $f_1 = [-3,4]$, $f_2(x) = x^2+2x$ and the domain of $f_2 = [-1,3]$, find: $(f_1+f_2)(x)$, $(f_1-f_2)(x)$, $\left(\frac{f_1}{f_2}\right)(x)$, $\left(\frac{f_2}{f_1}\right)(x)$ showing the domain of each function.

Solution:



$D_1 \cap D_2 = [-1,3]$

① $(f_1+f_2)(x) = x+2+x^2+2x = x^2+3x+2$

and the domain is $D_1 \cap D_2 = [-1,3]$

② $(f_1-f_2)(x) = x+2-(x^2+2x)$
$= x+2-x^2-2x = -x^2-x+2$

③ $\left(\frac{f_1}{f_2}\right)(x) = \frac{x+2}{x^2+2x} = \frac{x+2}{x(x+2)} = \frac{1}{x}$

, $Z(f_2) = \{0,-2\}$

The domain of $\left(\frac{f_1}{f_2}\right)$ is $D_1 \cap D_2 - Z(f_2)$

$D_1 \cap D_2 - Z(f_2) = [-1,3] - \{0\}$

Example:

Determine the domain of each of the Real functions defined by the rules:

[1] $f(x) = \sqrt{\dfrac{x-3}{x-5}}$ [2] $f(x) = \dfrac{\sqrt{x-3}}{\sqrt{x-5}}$

[3] $f(x) = \dfrac{\sqrt{x-3}}{\sqrt{5-x}}$

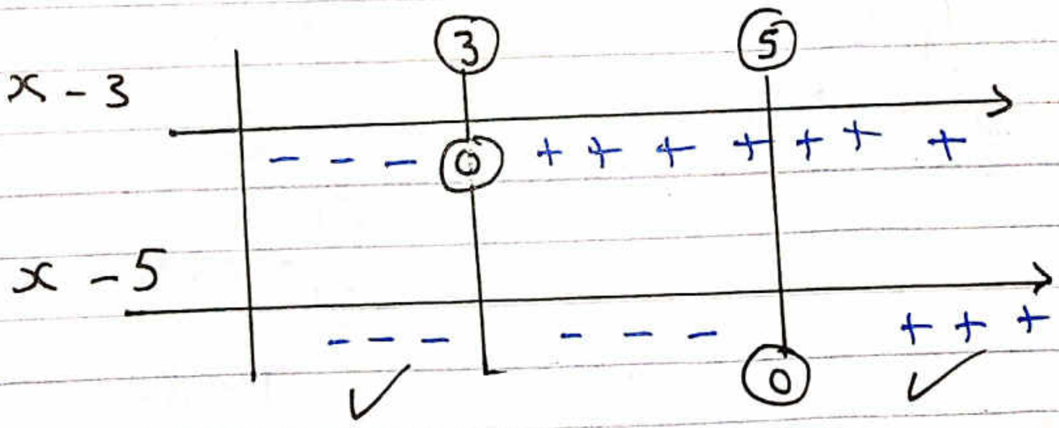
Solution

$\dfrac{x-3}{x-5} \geqslant 0$ where

$x-3 \geqslant 0 , x-5 > 0$

or

$x-3 \leqslant 0 , x < 0$



$\therefore$ The domain $= \mathbb{R} -]3, 5]$

② $f(x) = \dfrac{\sqrt{x-3}}{\sqrt{x-5}}$

Let $f(x) = \dfrac{f_1}{f_2}$

$D_1 = [3, \infty[$ domain of f_1

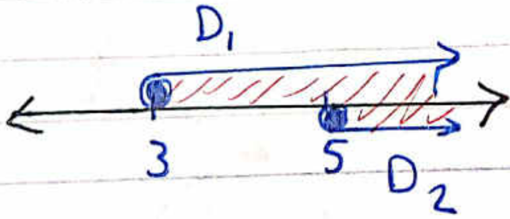
$D_2 =]5, \infty[$ domain of f_2

$Z(f_2) = \{5\}$

$\therefore$ the domain of $f(x)$

$= D_1 \cap D_2 - Z(f_2)$

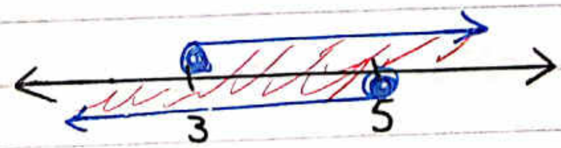
$=]5, \infty[$



3 $f(x) = \dfrac{\sqrt{x-3}}{\sqrt{5-x}}$

$D_1 = [3, \infty[$, $D_2 =]-\infty, 5]$

, $Z(f_2) = \{5\}$



$D = D_1 \cap D_2 - Z(f_2) = [3, 5[$

Second:

Composition of function

If f and g are two functions,

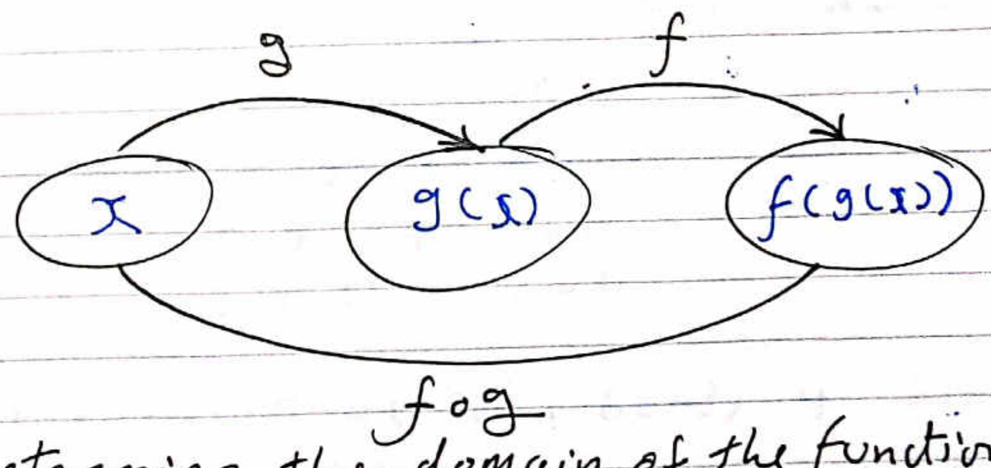
$$(f \circ g)(x) = f(g(x))$$

which read as (f composed g)

or (f after g)

* the domain of $(f \circ g)(x)$

$$= \{ x : x \in \text{domain of } g,\ g(x) \in \text{domain of } f \}$$



To determine the domain of the function $(f \circ g)$

(1) Find D_1 = domain of g

(2) Find D_2 = values of x that makes $g(x)$ in domain of f

(3) Find: $D_1 \cap D_2$ which is the domain of $(f \circ g)$

Example: If $f(x) = 3x+1$, $g(x) = x^2-5$

find:

(1) $(f \circ g)(2)$ ② $(g \circ f)(-3)$

<u>solution:</u>

1 $(f \circ g)(x) = f(g(x))$

$$= 3\,g(x)+1$$
$$= 3(x^2-5)+1$$
$$= 3x^2-15+1 = 3x^2-14$$

$\Rightarrow (f \circ g)(2) = 3(2)^2-14 = -2$

2 $(g \circ f)(x) = g(f(x))$

$$= (f(x))^2-5$$
$$= (3x+1)^2-5$$
$$= 9x^2+6x+1-5$$
$$= 9x^2+6x-4$$

$(g \circ f)(-3) = 9(-3)^2+6(-3)-4$

$= 59$

Scanned by CamScanner

Example: If $f(x) = \frac{1}{x}$, $g(x) = x+3$,

find (1) $(f \circ g)(x)$ (2) $(g \circ f)(x)$
and deduce the domain of each function.

Solution:

D_1 = domain of $g = \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = \frac{1}{g(x)} = \frac{1}{x+3}$$

$$g(x) \neq 0 \Rightarrow x+3 \neq 0 \Rightarrow x \neq -3$$

$$D_2 = \mathbb{R} - \{-3\}$$

$$D = D_1 \cap D_2 = \mathbb{R} \cap (\mathbb{R} - \{-3\}) = \mathbb{R} - \{-3\}$$

(2) D_1 = domain of $f = \mathbb{R} - \{0\}$

$$(g \circ f)(x) = g(f(x)) = f(x) + 3 = \frac{1}{x} + 3$$

the value of x which makes $f(x)$ in the domain of g is D_2

$$D_2 = \mathbb{R} - \{0\}$$

$$\therefore D = D_1 \cap D_2 = \mathbb{R} - \{0\}$$

20

Example: If $f(x) = x^2 - 3$, $g(x) = \sqrt{x-2}$ find $(f \circ g)(x)$ in the simplest form. , determine its domain, then find $(f \circ g)(3)$

Solution

D_1 = domain of $g = [2, \infty]$

$$(f \circ g)(x) = f(g(x)) = g(x)^2 - 3$$
$$= (\sqrt{x-2})^2 - 3$$
$$= x - 2 - 3$$
$$= x - 5$$

D_2 = the value of x which makes $g(x)$ in the domain of $f = \mathbb{R}$

the domain of $(f \circ g)(x) = D_1 \cap D_2$
$$= [2, \infty] \cap \mathbb{R}$$
$$= [2, \infty]$$

$(f \circ g)(3) = 3 - 5 = -2$

Example: If $f(x)=\sqrt{x+1}$, $g(x)=\frac{2}{x-3}$
then find the domain of $(f \circ g)$

Solution

D_1 = Domain of $g = \mathbb{R} - \{3\}$

the values of x which makes $g(x)$ in the domain of $f = D_2$

$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)+1}$

$$= \sqrt{\frac{2}{x-3}+1}$$

$$= \sqrt{\frac{x-3+2}{x-3}} = \sqrt{\frac{x-1}{x-3}}$$

$\Rightarrow \frac{x-1}{x-2} \geq 0$,

where $x-1 \geq 0, x-2 > 0$ or $x-1 \leq 0, x-2 < 0$

	1	3	
$x-1$	$- - -$ 0	$+ + +$	$+ + +$
$x-2$	$- - -$ ✓	$- -$	0 $+ + +$ ✓

$D_2 = \mathbb{R} -]1, 3]$

the domain of $(f \circ g) = D_1 \cap D_2 = \mathbb{R} -]1,3]$

If $f(x) = \sqrt{x-2}$, $g(x) = \sqrt{4-x}$

then find $f \circ g$

Solution:

$D_1 =$ Domain of $g =]-\infty, 4]$

$(f \circ g)(x) = f(g(x)) = \sqrt{g(x) - 2}$

$= \sqrt{\sqrt{4-x} - 2}$

$\because \sqrt{4-x} - 2 \geq 0$

$\Rightarrow \sqrt{4-x} \geq 2$ by squaring two sides

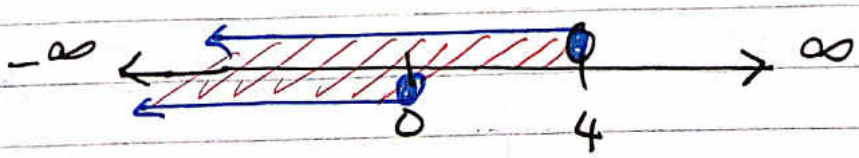
$4 - x \geq 4 \Rightarrow x \leq 0$

$\therefore$ the values of x which makes $g(x)$ in domain of f

$\therefore D_2 =]-\infty, 0]$

the domain of $(f \circ g) = D_1 \cap D_2$

$=]-\infty, 0]$



If $f(x) = \sqrt{x-1}$, $g(x) = \sqrt{x^2-4}$,

then find each of the following functions

(1) $f \circ g$ (2) $g \circ f$

Solution

$\boxed{1}$ $\because x^2 - 4 \geq 0$

$D_1 =$ Domain of g

$= \mathbb{R} -]-2, 2[$

$(f \circ g)(x) = f(g(x))$

$= \sqrt{\sqrt{x^2-4} - 1}$

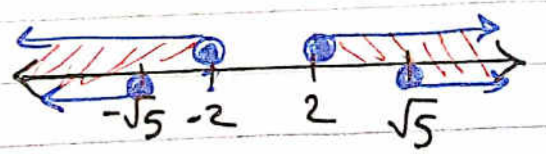
$\Rightarrow x^2 - 4 \geq 1$

$x^2 \geq 5$

$|x| \geq \sqrt{5}$

or $x^2 - 5 \geq 0$

$\Rightarrow D_2 = \mathbb{R} -]-\sqrt{5}, \sqrt{5}[$



$D = D_1 \cap D_2$

$= \mathbb{R} -]-\sqrt{5}, \sqrt{5}[$

$\boxed{2}$ $\because x - 1 \geq 0$

$\therefore D_1 =$ Domain of f

$= [1, \infty[$

$(g \circ f)(x) = g(f(x))$

$= \sqrt{(\sqrt{x-1})^2 - 4}$

$= \sqrt{x - 1 - 4}$

$= \sqrt{x - 5}$

$\because x - 5 \geq 0$

$D_2 = [5, \infty[$

$D = D_1 \cap D_2$

$= [5, \infty[$

Lesson 3:

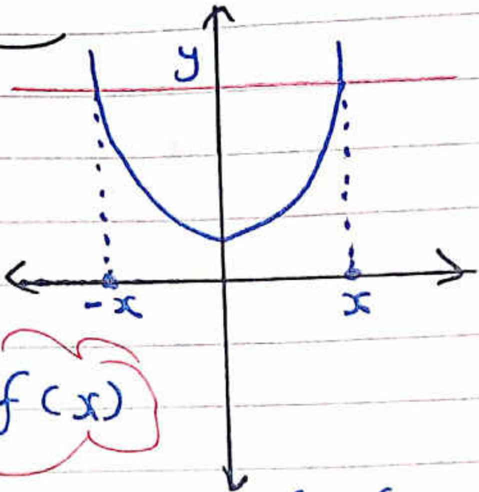
Some properties of functions

[I] Even and odd functions:

Even function:

The function f is said to be even if

$$f(-x) = f(x)$$



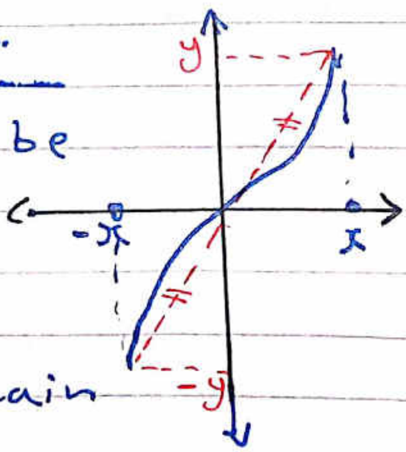
for each $x, -x \in$ the domain of f

note that

The curve of the even function is symmetry about y-axis.

Odd function

the function f is said to be odd if $f(-x) = -f(x)$



for each $x, -x \in$ the domain of f

note:

The curve of the odd function is symmetric about the origin point.

Example: Determine which of the functions defined by the following rules is even, which is is odd and which is neither even nor odd:

1) $f(x) = x^4 + x^2 - 1$

$\because$ the domain $= \mathbb{R}$ $\therefore x, -x \in \mathbb{R}$

, $f(-x) = (-x)^4 + (-x)^2 - 1$

$= x^4 + x^2 - 1 = f(x)$

$\therefore f$ is even.

2) $f(x) = 3x - 4x^3$

$\because$ The domain $= \mathbb{R}$ $\therefore x, -x \in \mathbb{R}$

$f(-x) = 3(-x) - 4(-x)^3$

$= -3x + 4x^3 = -(3x - 4x^3) = -f(x)$

$\therefore f$ is odd

3) $f(x) = \dfrac{x^2}{1+x}$

Solution: Domain $= \mathbb{R} - \{-1\}$

$\therefore$ for $x = 1 \in$ Domain there is no exist $-1 \in$ the domain

$\therefore f$ is neither even nor odd

Another Solution:

$f(-x) = \dfrac{(-x)^2}{1+(-x)} = \dfrac{x^2}{1-x} = -\dfrac{x^2}{-1+x}$

$\therefore f(-x) \neq f(x)$, $f(-x) \neq -f(x)$

④ $f(x) = \sqrt{x+3}$

Solution: Domain $= [-3, \infty[$

for each $x \in [-3, \infty[$
it is not necessary to find $-x \in [-3, \infty[$
for example $4 \in [-3, \infty[$, $-4 \notin [-3, \infty[$

$\therefore f$ is neither even nor odd.

⑤ $f(x) = (x^2+1)^3$

Solution Domain $= \mathbb{R}$

$f(-x) = ((-x)^2+1)^3 = (x^2+1)^3 = f(x)$

$\therefore f$ is even function.

⑥ $f(x) = \sqrt{x^2+6}$

Solution: Domain $= \mathbb{R}$

$f(-x) = \sqrt{(-x)^2+6} = \sqrt{x^2+6} = f(x)$

$\therefore f$ is even function

(7) $f(x) = x^3 - \frac{1}{x}$

Solution: Domain $= \mathbb{R} - \{0\}$

$$f(-x) = (-x)^3 - \frac{1}{-x} = -x^3 + \frac{1}{x}$$

$$= -\left(x^3 - \frac{1}{x}\right) = -f(x)$$

f is odd function

(8) $f(x) = x \cos x$

solution:

$$f(-x) = (-x) \cos -x$$

$$= -x \cos x$$

$$= -f(x)$$

f is an odd function

$\cos(-x) = \cos x$

$\sin(-x) = -\sin x$

$\tan(-x) = -\tan x$

(9) $f(x) = \begin{cases} 2x + x^2 & , x \leq 0 \\ 2x - x^2 & , x > 0 \end{cases}$

Solution:

$$f(-x) = \begin{cases} 2(-x) + (-x)^2, & -x \leq 0 \\ 2(-x) - (-x)^2, & -x > 0 \end{cases}$$

$$= \begin{cases} -2x + x^2 & , x \geq 0 \\ -2x - x^2 & , x < 0 \end{cases}$$

$$= -\begin{cases} 2x - x^2 & , x \geq 0 \\ 2x + x^2 & , x < 0 \end{cases}$$

$$= -\begin{cases} 2x + x^2, & x \leq 0 \\ 2x - x^2 & , x > 0 \end{cases} \Rightarrow \text{is odd}$$

$$= -f(x)$$

one-to-one function

the function $f : X \to Y$ is called one-to-one function if for each $a, b \in X$, $\boxed{f(a) = f(b)}$ then $\boxed{a = b}$

the horizontal line test:

If there exist a horizontal line intersects the curve of the function at more than one point, then the curve represents a function is not one-to-one

notice that

the even functions, in general are not one-to-one

prove that the functions that are defined by the following rules are one-to-one:

[1] $f(x) = 2x - 3$ [2] $f(x) = \dfrac{3x-5}{4x+3}$

Solution:

[1] let $a, b \in$ the domain of f
put $f(a) = f(b)$

$\Rightarrow 2a - \cancel{3} = 2b - \cancel{3}$

$\Rightarrow \cancel{2}a = \cancel{2}b$

$\Rightarrow a = b \quad \therefore f$ is one-to-one

[2] let $a, b \in$ the domain of f put $f(a) = f(b)$

$$\Rightarrow \frac{3a-5}{4a+3} = \frac{3b-5}{4b+3}$$

$\Rightarrow (3a-5)(4b+3) = (3b-5)(4a+3)$

$\Rightarrow 12ab + 9a - 20b - 15 = 12ab + 9b - 20a - 15$

$\Rightarrow 9a - 20b = 9b - 20a$

$\Rightarrow 29a = 29b$

$\Rightarrow a = b$

$\therefore f$ is one-to-one

prove that functions that are defined by the following rules are not one-to-one:

(1) $f(x) = x^2$ (2) $f(x) = x^2 - 5x + 6$

Solution:

let $a, b \in$ Domain of f, let $f(a) = f(b)$

$\Rightarrow a^2 = b^2 \Rightarrow a = \pm b$

$\therefore$ a has two values $b, -b$

$\therefore f$ is not one-to-one.

(2) Let $a, b \in$ the domain of the function f

and put $f(a) = f(b)$

$\Rightarrow a^2 - 5a + 6 = b^2 - 5b + 6$

$\Rightarrow a^2 - 5a = b^2 - 5b$

$\Rightarrow a^2 - b^2 - 5a + 5b = 0$

$\Rightarrow (a-b)(a+b) - 5(a-b) = 0$

$\Rightarrow (a-b)(a+b-5) = 0$

$\Rightarrow a = b \quad | \quad a = -b + 5$

$\therefore$ a has two values $b, -b+5$

Graph the function f :

$$f(x) = \begin{cases} x+2 & , x \geq 0 \\ 2-x & , x < 0 \end{cases}$$

then from the graph:

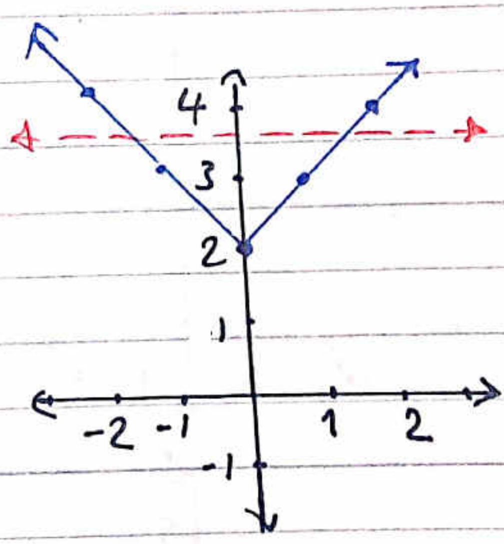
(1) Find the range of the function f

(2) Determine whether f is even, odd or otherwise

(3) Mention whether f is one-to-one or not giving reason.

Solution:

$x+2$			$2-x$		
0	1	2	0	-1	-2
2	3	4	2	3	4



from the graph:

(1) the range $= [2, \infty[$

(2) The function is even because it's symmetric about y-axis

(3) the function f is not one-to-one because there exist a horizontal line intersects the curve of the function at two points

Graph the function f:

$$f(x) = \begin{cases} 2 & \text{when } x > 0 \\ -2 & \text{when } x < 0 \end{cases}$$

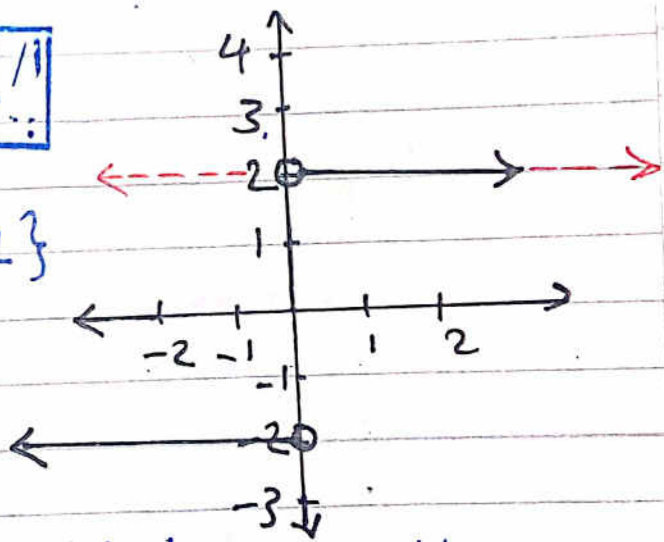
then from the graph:

(1) Find the range

(2) Determine whether f is even, odd or otherwise.

(3) Mention whether f is one-to-one or not giving reason

Solution



(1) The range $= \{-2, 2\}$

(2) the function is odd because the graph of the function is symmetric about the origin

(3) The function is not one-to-one because there exist a horizontal line intersects the curve of the function at more that one point

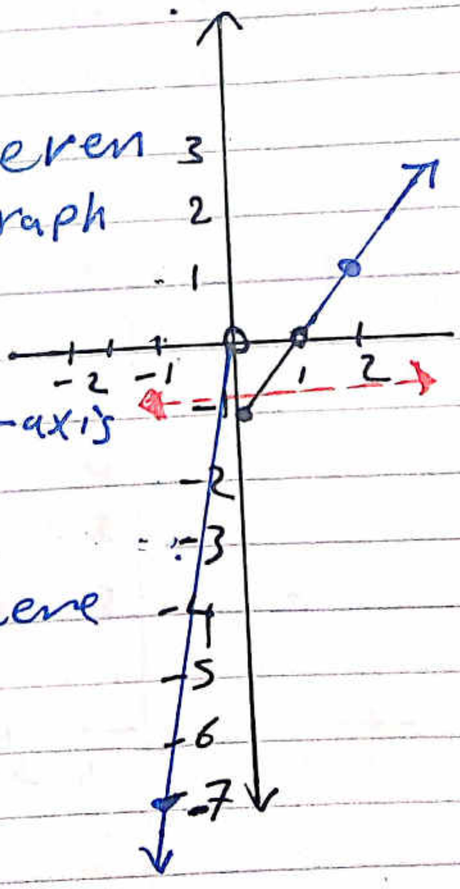
$$f(x) = \begin{cases} x-1 & , x \geq 0 \\ 7x & , x < 0 \end{cases}$$

(1) Find the range
(2) Determine whether f is even, odd or otherwise
(3) Menstion whether f is one-to-one or not giving reason

Solution:

$x-1$			$7x$		
0	1	2	0	-1	-2
-1	0	1	0	-7	-14

(1) the range $= \mathbb{R}$



(2) the function neither even nor odd because the graph of the function neither symmetric about origin nor symmetric about y-axis

(3) the function is not one-to-one because there exist a horizontal line intersects the curve at two points

Creative thinking:

Represent graphically the curve which satisfies each of the following conditions:
Passes through the points $(0,-2)$, $(2,2)$, $(3,7)$ and represents even function

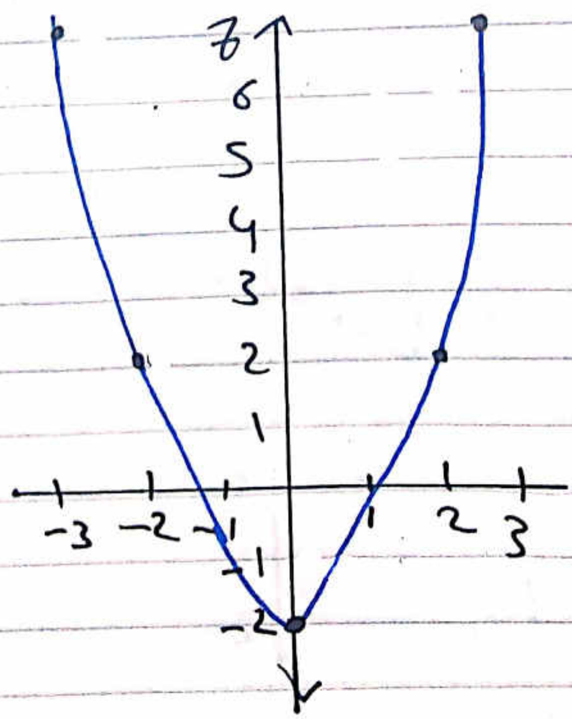
Solution

$\because$ the function is even

$\Rightarrow f(2) = f(-2) = 2$, $f(3) = f(-3) = 7$

$\therefore$ the curve of the function passes through the two points $(-2,2)$, $(-3,7)$ also

-3	-2	0	2	3
7	2	-2	2	7



Represent graphically the curve which satisfies each of the following conditions: passes through the points $(0,0)$, $(-2,1)$, $(-3,5)$ and represents an odd function.

Solution

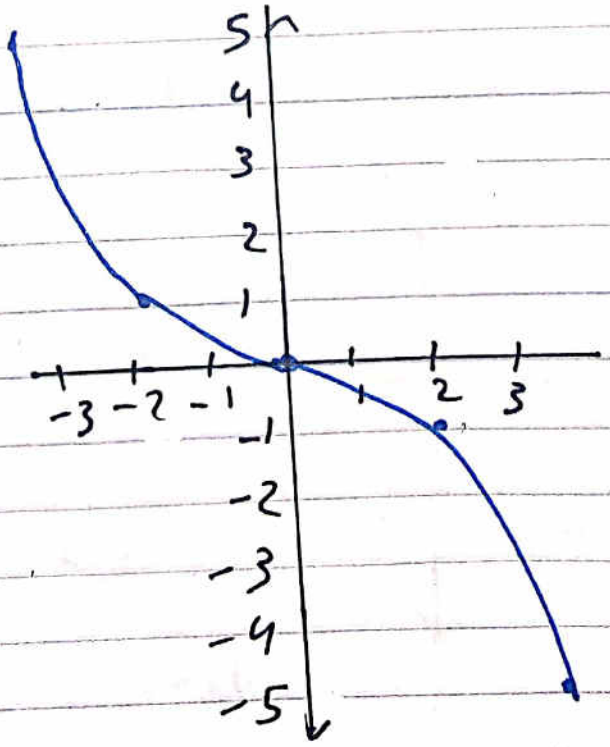
$\because$ the function is odd

$\therefore f(-2) = -f(2) = 1 \Rightarrow f(2) = -1$

, $f(-3) = -f(3) = 5 \Rightarrow f(3) = -5$

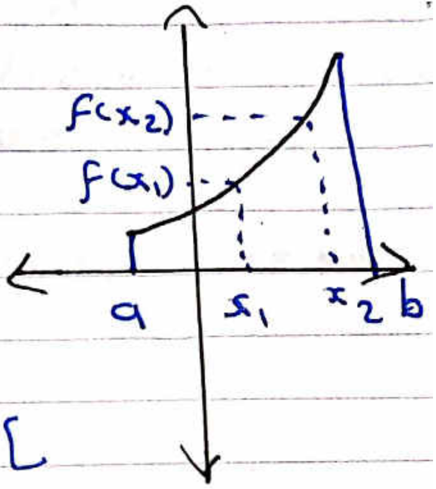
$\Rightarrow$

-3	-2	0	2	3
5	1	0	-1	-5



Lesson 4: Monotony of functions

[1] Increasing function:



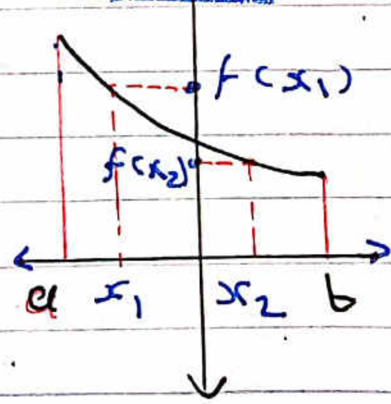
f is said to be increasing on $]a, b[$ if:

$$x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$$

for every $x_1, x_2 \in]a, b[$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٢٩٢٦٦٨٧

[2] Decreasing function:

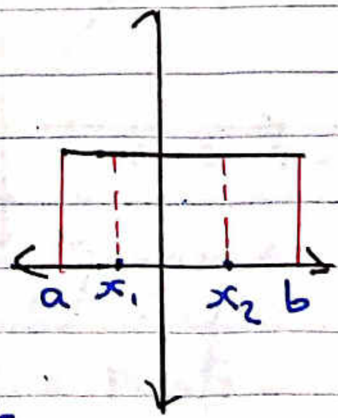


f is said to be decreasing on $]a, b[$ if:

$$x_2 > x_1 \Rightarrow f(x_1) > f(x_2)$$

for every $x_1, x_2 \in]a, b[$

[3] Constant function:



f is said to be constant on $]a, b[$ if:

$$x_2 > x_1 , f(x_2) = f(x_1)$$

for every $x_1, x_2 \in]a, b[$

If $f : [-2, 6] \to \mathbb{R}$ where

$$f(x) = \begin{cases} 4 - x & , -2 \le x < 1 \\ x & , 1 \le x \le 6 \end{cases}$$

(1) Graph the function f, and from the graph deduce its range, and discuss its monotonicity

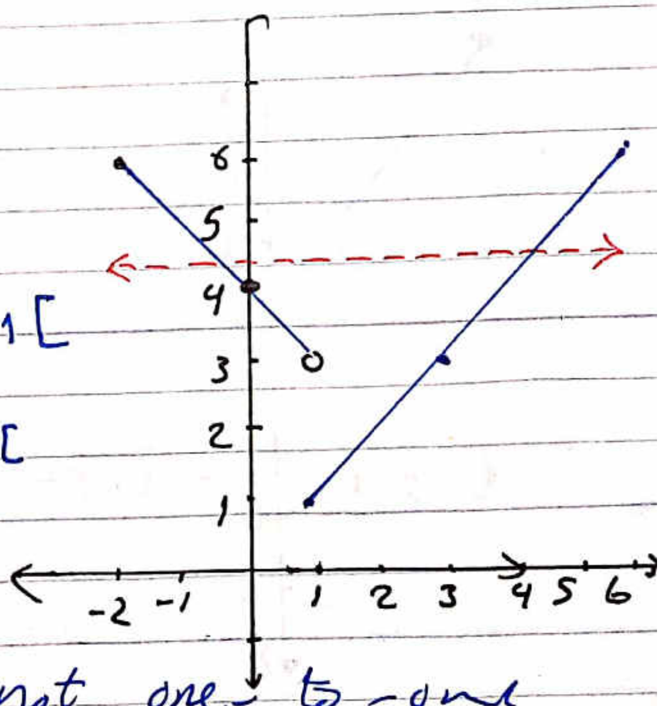
(2) Is f one-to-one? Explain your answer

أ/ أحمد عمر
معلم أول رياضيات

Solution:

$f_1(x) = 4 - x$ | $f_2(x) = x$

-2	0	1	1	3	6
6	4	3	1	3	6



(1) The range $= [1, 6]$

The function is

- decreasing on $]-2, 1[$
- increasing on $]1, 6[$

(2) the function is not one-to-one Because there exist a horizontal line intersects the function at two points

If $f_1 : \mathbb{R} \to \mathbb{R}$ where $f_1(x) = 3x - 1$, (38)

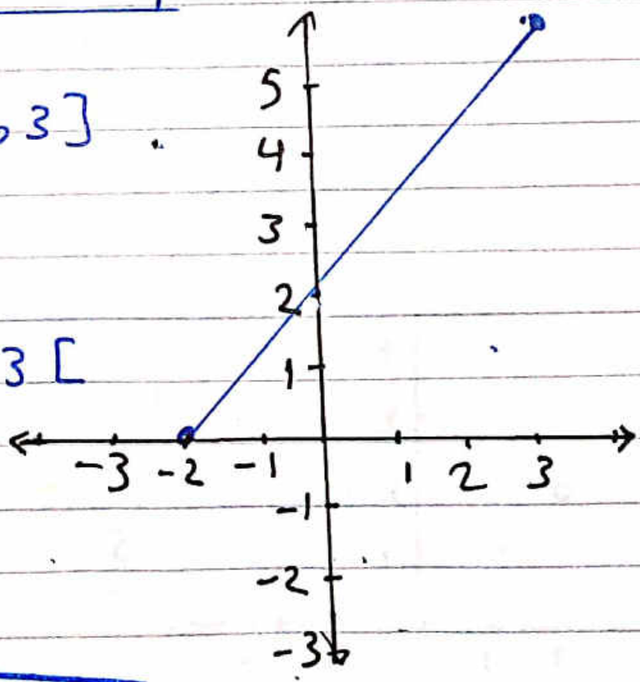
$f_2 : [-2, 3] \to \mathbb{R}$ where $f_2(x) = 3 - 2x$ graph the function $(f_1 + f_2)$ showing its domain, then deduce its monotonicity.

Solution

$D_1 \cap D_2 = \mathbb{R} \cap [-2,3] = [-2,3]$

$(f_1 + f_2)(x) = 3x - 1 + 3 - 2x = x + 2$

x	-2	0	3
$f(x)$	0	2	5



the domain $= [-2,3]$.

the function is

increasing on $]-2,3[$

If $f(x) = x^3 - 4x$, $g(x) = x^2 - 4$, determine (32)
the domain of the function $\frac{f}{g}$ and
graph it,
from the graph determine its range and
mention whether it is even, odd or otherwise
and discuss its monotonicity, and mention
it is one-to-one or not.

Solution:

$D_1 = \mathbb{R}$, $D_2 = \mathbb{R}$, $D_1 \cap D_2 = \mathbb{R}$

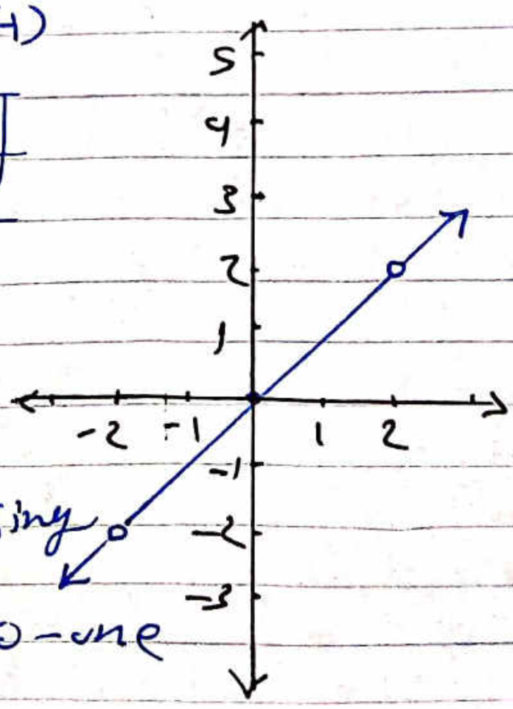
$Z(g) = \{-2, 2\}$

Domain of $(\frac{f}{g}) = D_1 \cap D_2 - Z(g)$

$= \mathbb{R} - \{-2, 2\}$

$(\frac{f}{g})(x) = \frac{x^3 - 4x}{x^2 - 4} = \frac{x(x^2 - 4)}{(x^2 - 4)} = x$

-2	0	2
-2	0	2



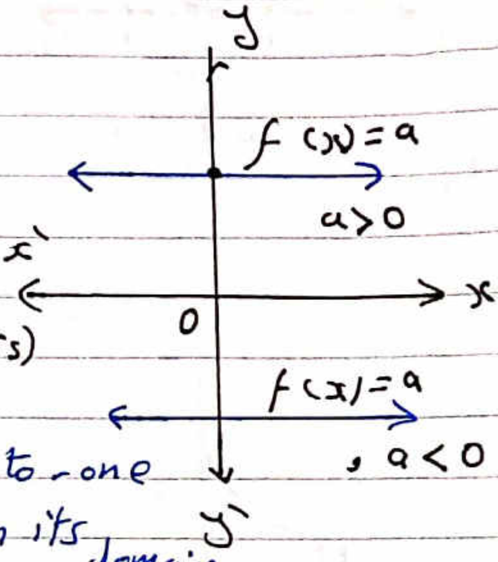
- The range $= \mathbb{R} - \{-2, 2\}$
- the function is odd
- the function is increasing on $\mathbb{R} - \{-2, 2\}$
- the function is one-to-one

Scanned by CamScanner

Lesson 5: Graphical representation of functions and geometrical transformations

[1] The Constant function:

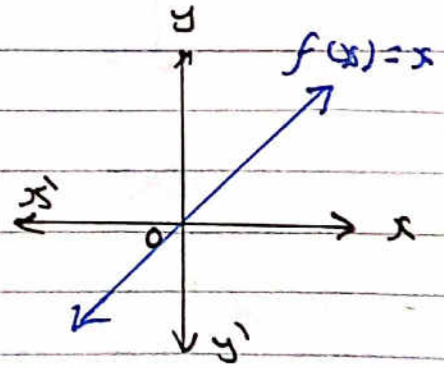
$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = a$



- the range = $\{a\}$
- The function is even (symmetric about y-axis)
- the function is not one-to-one
- the function is Constant on its domain

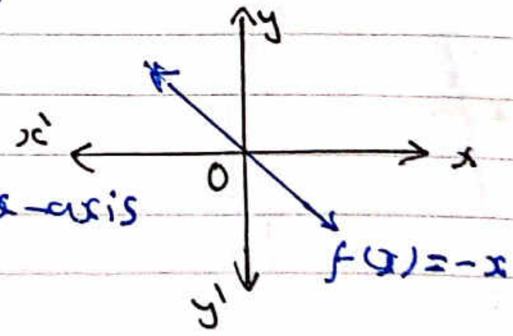
[2] The linear function:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x$



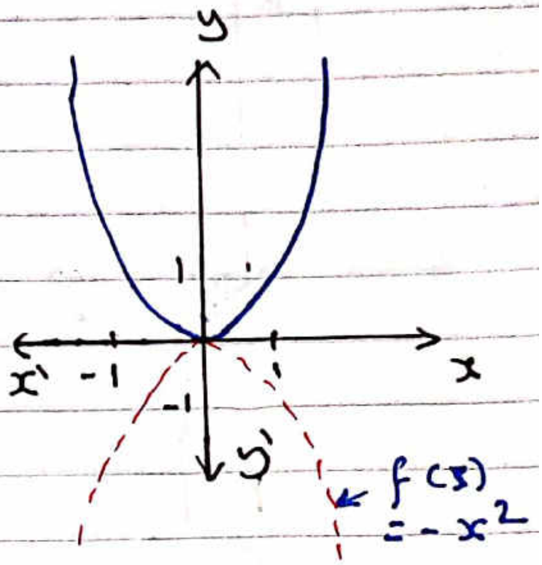
- range = $\mathbb{R}$
- the function is odd (symmetric about the origin)
- the function is one-to-one
- the function is increasing on its domain $\mathbb{R}$

$f(x) = -x$ is reflection of $f(x) = x$ about x-axis



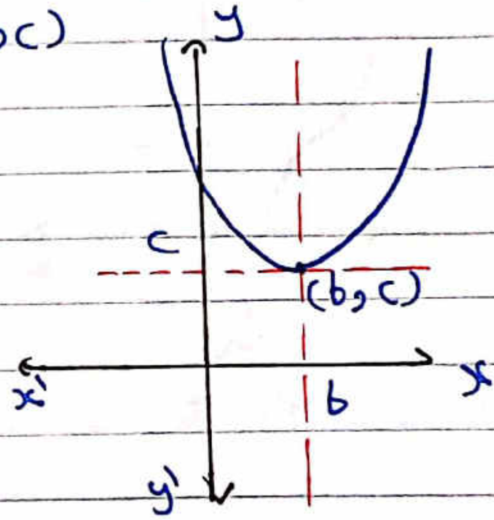
(3) The quadratic function:

$f: \mathbb{R} \to \mathbb{R}, \; f(x) = x^2$

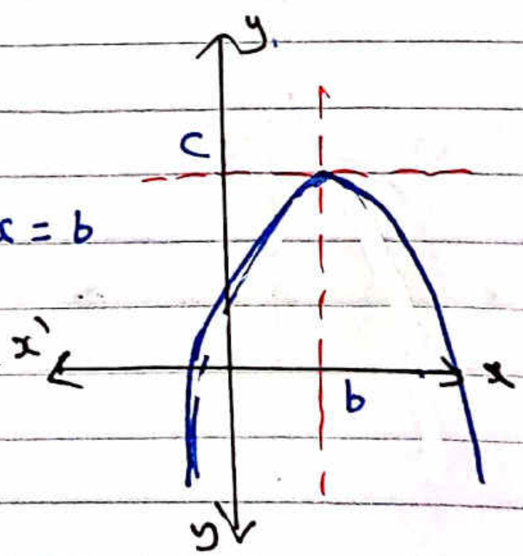


* Range = $[0, \infty[$
* the function is
- even
- decreasing on $]-\infty, 0[$
- increasing on $]0, \infty[$
- not one-to-one

* $f(x) = a(x-b)^2 + c$
- point of vertex is (b, c)
- Range $= [c, \infty[$
- decreasing on $]-\infty, b[$
- increasing on $]b, \infty[$
- not one-to-one
- neither even nor odd
  if $b = 0$ it is even



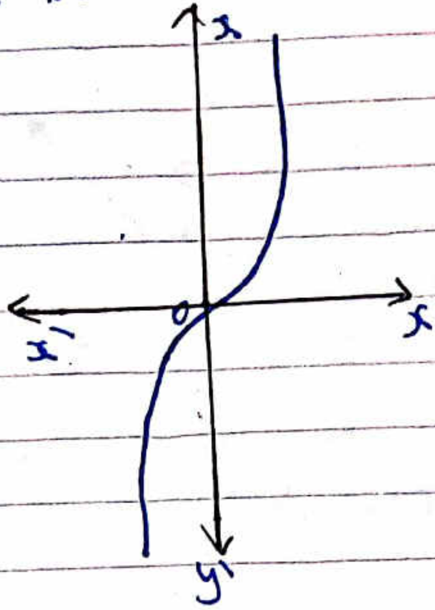
$f(x) = -a(x-b)^2 + c$
- increasing on $]-\infty, b[$
- decreasing on $]b, \infty[$
- axis of symmetry is $x = b$
- Range $=]-\infty, c]$



$\boxed{3}$ The Cubic function:

$f: \mathbb{R} \to \mathbb{R},\ f(x) = x^3$

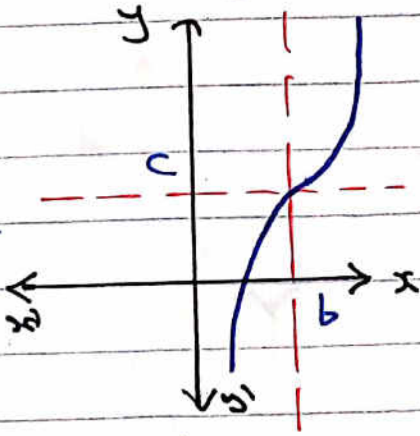
* Range $= \mathbb{R}$
* the function is
  - odd
  - one-to-one
  - increasing on its domain $\mathbb{R}$



$f(x) = a(x-b)^3 + c$

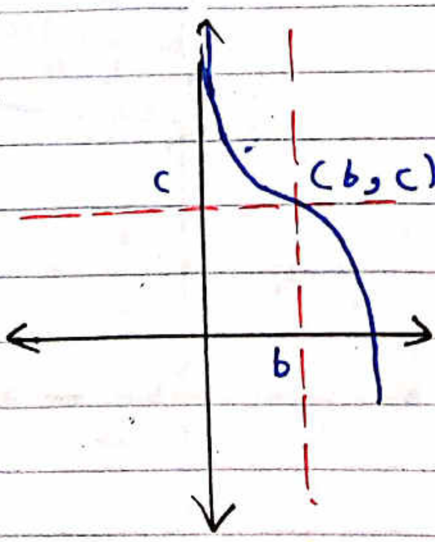
point of symmetry (b, c)

- Range $= \mathbb{R}$
- the function is one-to-one
- the function neither even nor odd except if $b = 0$, $c = 0$, then, it is odd
- The function is increasing about its domain

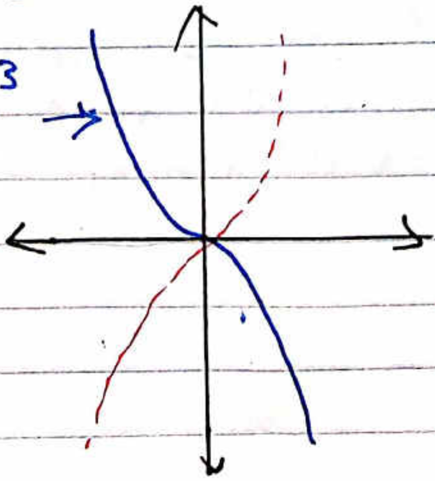


* $f(x) = -a(x-b)^3 + c$

the function is decreasing on its domain



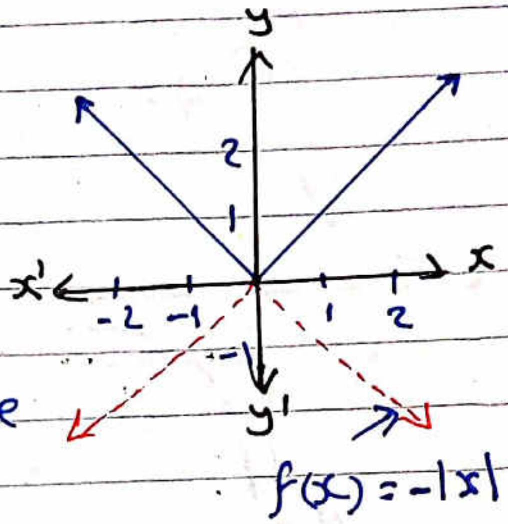
$f(x) = -x^3 \rightarrow$



[4] Absolute function:

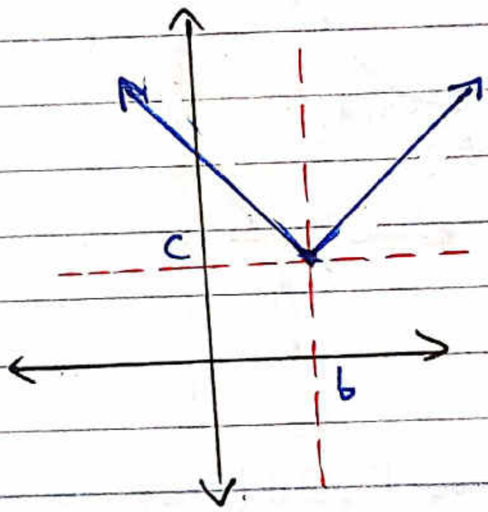
$f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$

* $f(x) = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$



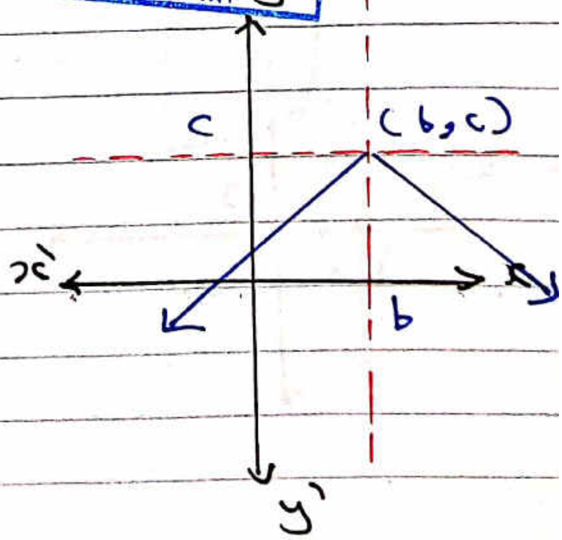
* Range = $[0, \infty[$
* the function is:
- even , not one-to-one
- increasing on $]0, \infty[$
- decreasing on $]-\infty, 0[$

* $f(x) = a|x-b| + c$
- Range = $[c, \infty[$
- increasing on $]b, \infty[$
- decreasing on $]-\infty, b[$
- not one-to-one
- neither even nor odd
  if $b = 0$ it is even



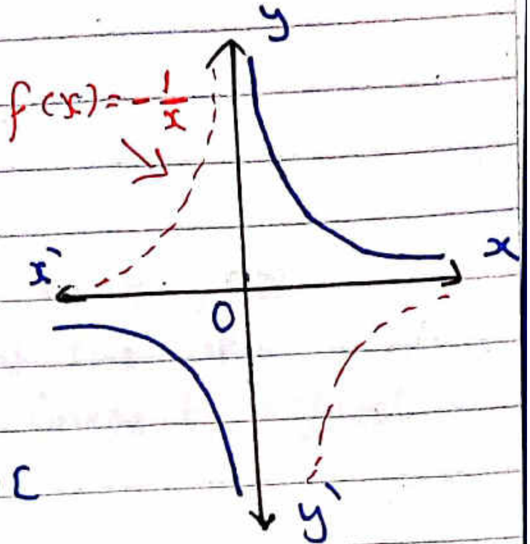
* $f(x) = -a|x-b| + c$

- Range = $]-\infty, c]$
- increasing on $]-\infty, b[$
- decreasing on $]b, \infty[$
- axis of symmetry is $x = b$



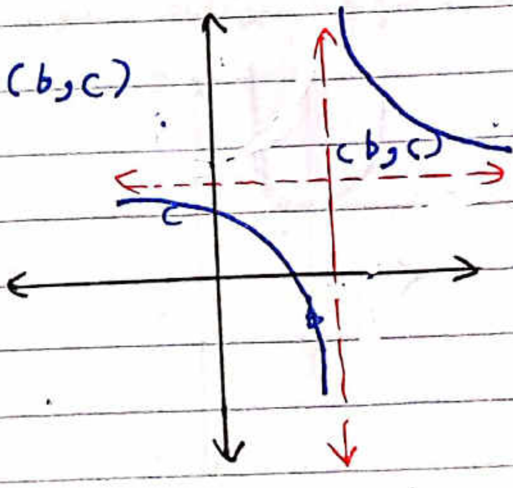
5 Rational function:

$f: \mathbb{R} - \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$



* Range $= \mathbb{R} - \{0\}$
* The function is
  - odd
  - one-to-one
  - decreasing on $]-\infty, 0[$ and decreasing on $]0, \infty[$

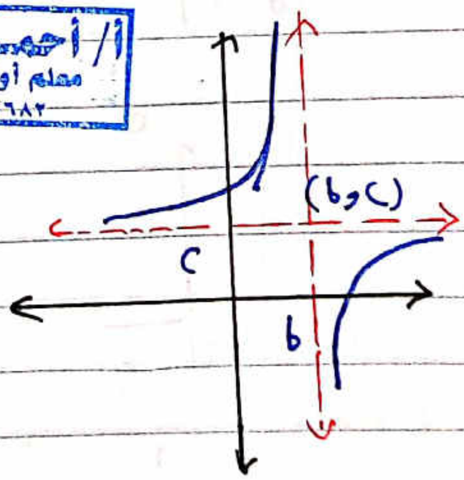
- $f(x) = \frac{a}{x-b} + c$



* point of symmetry is (b, c)
* Domain $= \mathbb{R} - \{b\}$
* Range $= \mathbb{R} - \{c\}$
* the function is
  - one-to-one
  - neither even nor odd

    if $b = 0$, $c = 0$ it is odd
  - the function is decreasing on $]-\infty, b[$, $]b, \infty[$

- $f(x) = \frac{-a}{x-b} + c$



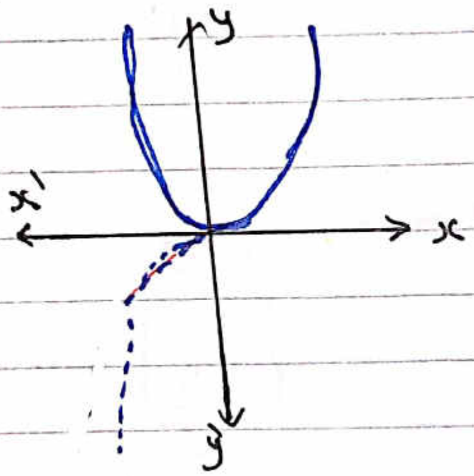
- Range $= \mathbb{R} - \{c\}$
- the function is increasing on $]-\infty, b[$, $]b, \infty[$

* the curve of $y = |f(x)|$

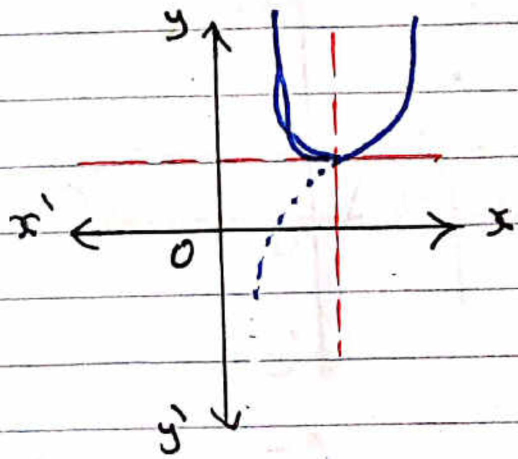
$$y = \begin{cases} f(x) & , f(x) \geq 0 \\ -f(x) & , f(x) < 0 \end{cases}$$

Is represented by the curve of $y = f(x)$ with replacing the part of the curve which is under the x-axis by its image by reflection in the x-axis

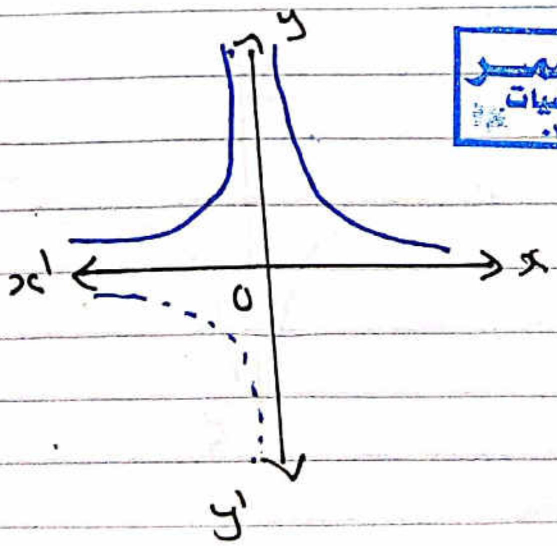
- $f(x) = |x^3|$



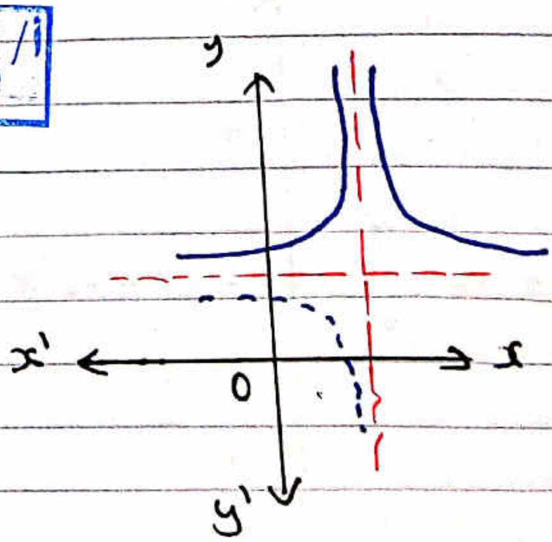
$f(x) = a|(x-b)^3| + c$



- $f(x) = \dfrac{1}{|x|}$



$f(x) = \dfrac{a}{|x-b|} + c$

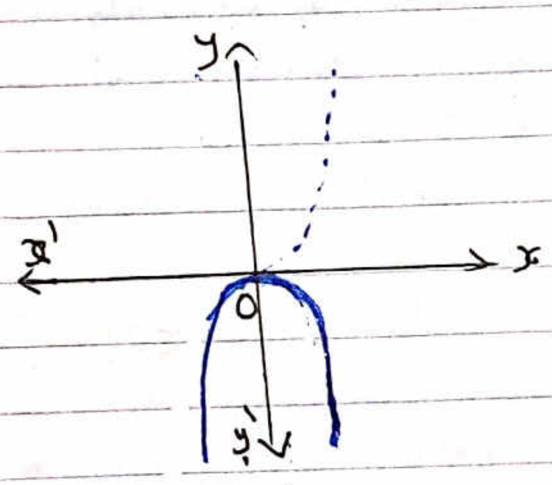


The curve of $y = -|f(x)|$

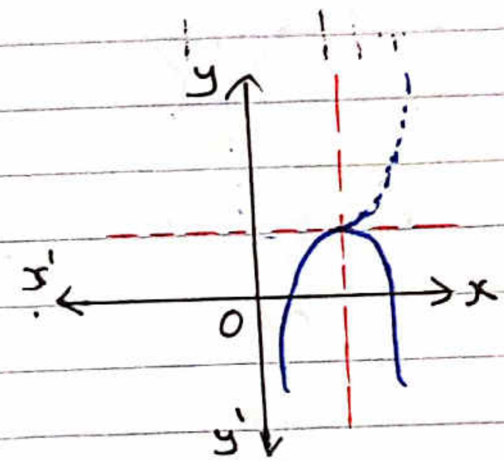
$$y = \begin{cases} -f(x) & , f(x) \geq 0 \\ f(x) & , f(x) < 0 \end{cases}$$

Is represented by the curve of $y = f(x)$ with replacing the part of the curve wich is up the x-axis by its image by reflection in the x-axis.

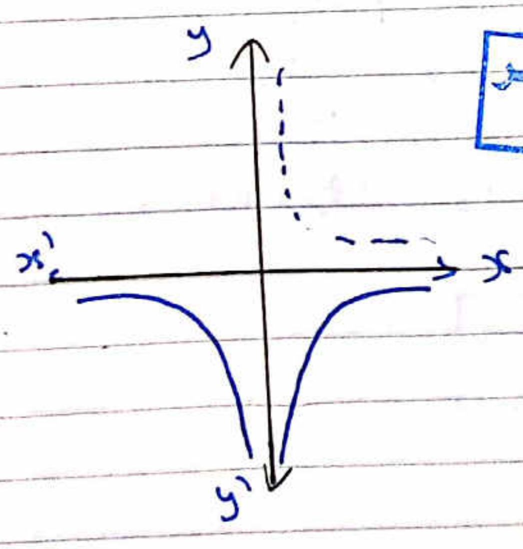
- $f(x) = -|x^3|$



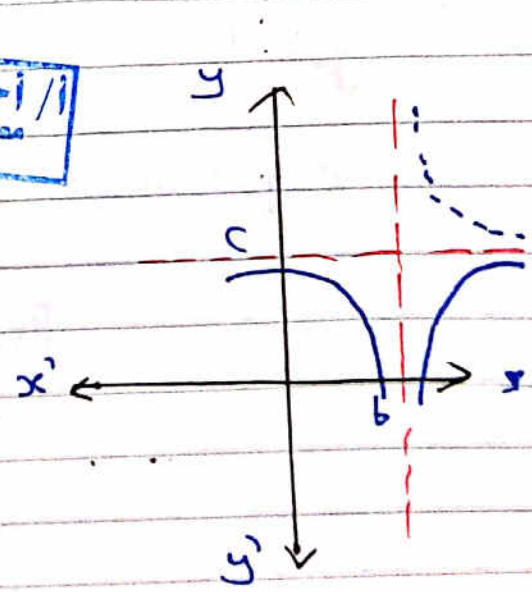
$f(x) = -a|(x-b)^3| + c$



- $f(x) = \dfrac{-1}{|x|}$



$f(x) = \dfrac{-a}{|x-b|} + c$



(47)

Use the curve of the function f where $f(x) = x^2$ to represent each of the following functions, from the graph find the domain and the range and discuss the monotonicity and its type whether it is even odd or otherwise

① $g(x) = x^2 - 3$

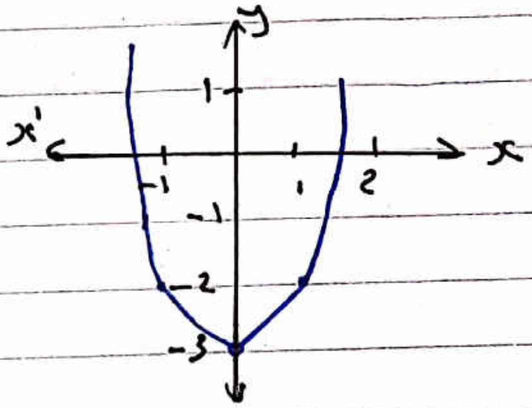
② $g(x) = 2 - x^2$

③ $g(x) = (x-2)^2$

③ $g(x) = (x+2)^2 - 4$

Solution

1 $g(x) = x^2 - 3$

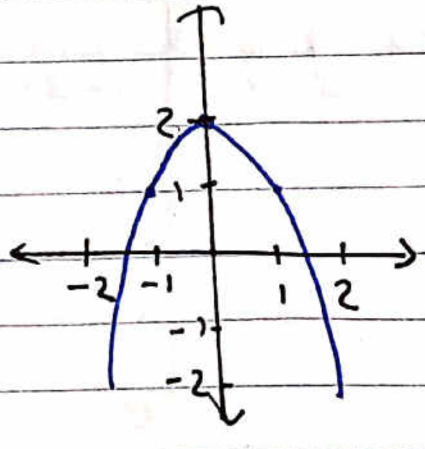


point of vertex is $(0, -3)$

- range $= [-3, \infty[$
- decreasing on $]-\infty, 0[$
- increasing on $]0, \infty[$
- the function is even
- the equation of axis of symmetry is $x = 0$

2 $g(x) = 2 - x^2$

$g(x) = -x^2 + 2$



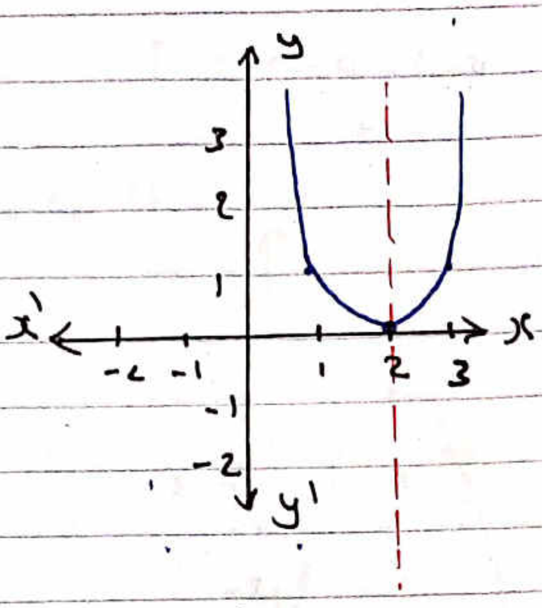
point of the vertex is $(0, 2)$

- range $=]-\infty, 2]$
- increasing on $]-\infty, 0[$
- decreasing on $]0, \infty[$
- the function is even
- the equation of axis of symmetry is $x = 0$

3) $g(x) = (x-2)^2$

point of the vertex is $(2, 0)$

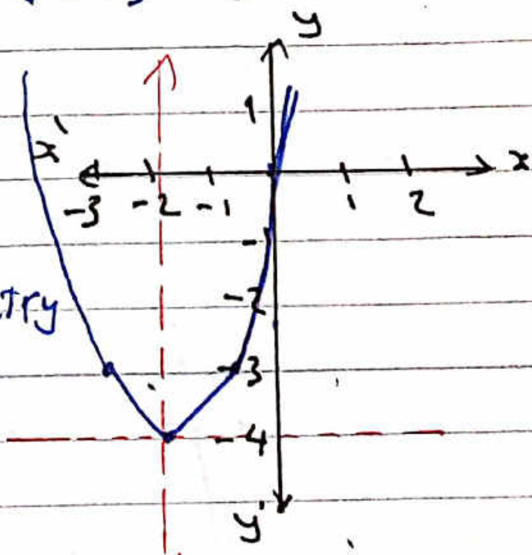
- Range $= [0, \infty[$
- decreasing on $]-\infty, 2[$
- increasing on $]2, \infty[$
- neither even nor odd
- equation of axis of symmetry is $x = 2$



4) $g(x) = (x+2)^2 - 4$

point of symmetry is $(-2, -4)$

- Range $= [-4, \infty[$
- decreasing on $]-\infty, -2[$
- increasing on $]-2, \infty[$
- neither even nor odd
- equation of axis of symmetry is $x = -2$

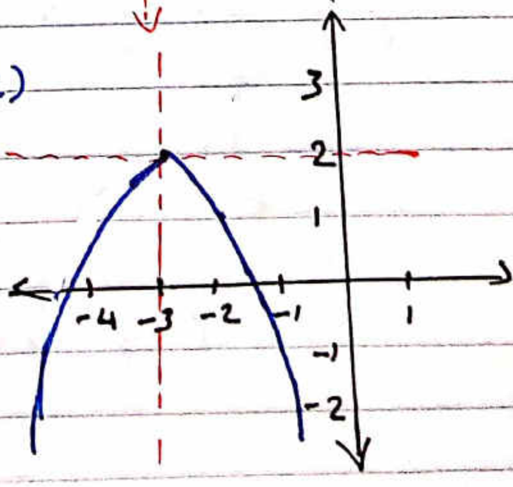


5) $g(x) = 2 - (x+3)^2$

$g(x) = -(x+3)^2 + 2$

point of symmetry is $(-3, 2)$

- Range $=]-\infty, 2]$
- increasing on $]-\infty, -3[$
- decreasing on $]-3, \infty[$
- neither even nor odd
- equation of axis of symmetry is $x = -3$



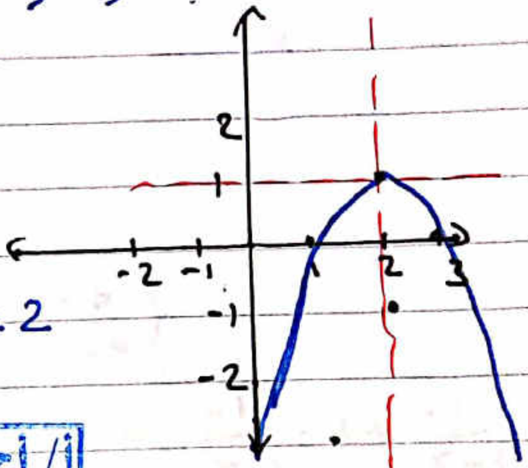
(6) $g(x) = 4x - x^2 - 3$

$$g(x) = -x^2 + 4x - 3$$
$$= -(x^2 - 4x + 3)$$
$$= -[(x-2)^2 - 1]$$
$$= -(x-2)^2 + 1$$

$$x = \frac{-b}{2a} = \frac{-4}{-2} = 2$$

$$f(2) = -4 + 8 - 3$$
$$= 1$$

point of symmetry is $(2, 1)$

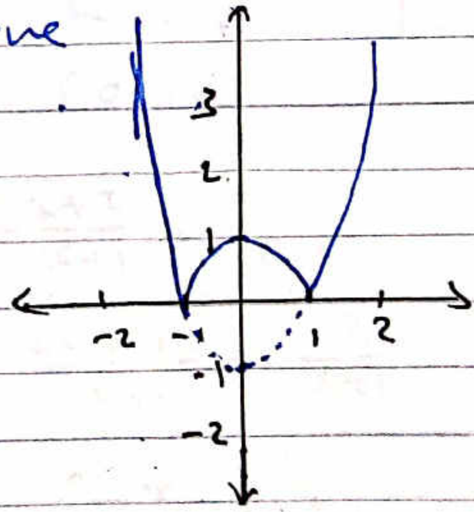


- Range $=]-\infty, 1]$
- increasing on $]-\infty, 2[$
- decreasing on $]2, \infty[$
- neither even nor odd
- axis of symmetry $x = 2$

(7) $g(x) = |x^2 - 1|$

let $f(x) = x^2 - 1$
point of symmetry is $(0, -1)$

replacing the part of the curve wich is under the x-axis by its image by reflection in the x-axis



- Range $= [0, \infty[$
- decreasing on $]-\infty, -1[$
- decreasing on $]0, 1[$
- increasing on $]-1, 0[$
- increasing on $]1, \infty[$
- equation of axis of symmetry is $x = 0$

Notes

1. $|a-x| = |x-a|$

2. $(a-x)^2 = (x-a)^2$

3. $(a-x)^3 = -(x-a)^3$

4. $f(x) = ax^2+bx+c$

$$= a\left[x-\left(\frac{-b}{2a}\right)\right]^2 + f\left(\frac{-b}{2a}\right)$$

5. $\Rightarrow \dfrac{x-3}{x-2} = \dfrac{x-2-1}{x-2} = \dfrac{x-2}{x-2} + \dfrac{-1}{x-1} = 1 + \dfrac{-1}{x-1}$

point of symmetry is $(1,1)$

$\Rightarrow \dfrac{x+1}{x} = \dfrac{x}{x} + \dfrac{1}{x} = 1 + \dfrac{1}{x}$

point of symmetry is $(0,1)$

$\Rightarrow \dfrac{2x}{x+1} = \dfrac{2x+2-2}{x+1} = \dfrac{2x+2}{x+1} + \dfrac{-2}{x+1}$

$= \dfrac{2(x+1)}{x+1} + \dfrac{-2}{x+1}$

$= 2 + \dfrac{-2}{x+1}$

point of symmetry is $(-1,2)$

Lesson 6: Solving absolute value equations and inequalities

Example [1] Find algebraically in $\mathbb{R}$ the solution set of each of the following equations:

[1] $|x| = 5$

$\because |x| = 5 \quad \therefore x = 5$ or $x = -5$

$\therefore S.S = \{5, -5\}$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٢٣٦٢٣٦٦٨٢

[2] $|3 - 2x| = 7$

Solution:

$\because |2x - 3| = 7 \quad \therefore 2x - 3 = \pm 7$

$x \geq \frac{3}{2}$	$x < \frac{3}{2}$
$2x - 3 = 7$	$2x - 3 = -7$
$2x = 10$	$2x = -4$
$x = 5 \geq \frac{3}{2}$ ✓	$x = -2 < \frac{3}{2}$ ✓

$S.S = \{-2, 5\}$

[3] $|x - 2| = 3x - 4$

Solution

$x \geq 2$	$x < 2$
	$x - 2 = -(3x - 4)$
$x - 2 = 3x - 4$	$x - 2 = -3x + 4$
$2x = 2 \quad \div 2$	$4x = 6 \quad \div 4$
$x = 1 \not\geq 2$ ✗	$x = \frac{3}{2} < 2$ ✓
(refused)	

$S.S = \{\frac{3}{2}\}$

$\boxed{4}$ $|x+2| + x - 2 = 0$

Solution:

$$|x+2| = -x+2$$

$x \geq -2$	$x < -2$
$x+2 = -x+2$	$x+2 = -(-x+2)$
$2x = 0$	$x+2 = x-2$
$x = 0 \geq -2$ ✓	$\Rightarrow 2 = -2$
	(this is impossible)

$\boxed{5}$ $x + |x| = 2$

Solution:

$$|x| = -x+2$$

$x \geq 0$	$x < 0$
$x = -x+2$	$x = -(-x+2)$
$2x = 2$	$x = x-2$
$x = 1 \geq 0$ ✓	$\Rightarrow 0 = -2$ ✗
	(not possible)

$S.S = \{1\}$

$\boxed{6}$ $|x+2| - x + 1 = 0$

Solution

$$|x+2| = x-1$$

$x \geq -2$	$x < -2$
$x+2 = x-1$	$x+2 = -x+1$
$2 = -1$ (✗)	$2x = -1$
	$\Rightarrow x = -\frac{1}{2} \not< -2$ (✗)

$S.S = \phi$

(7) $|2x-6| = |x-3|$

Solution:

$$2x-6 = \pm(x-3)$$

$x \geq 3$

$$2x-6 = x-3$$

$$\boxed{x = 3} \checkmark$$

$x < 3$

$$2x-6 = -x+3$$

$$3x = 9$$

$$\boxed{x = 3}$$

$$S.S = \{3\}$$

another solution

$$|2(x-3)| = |x-3|$$

$$2|x-3| - |x-3| = 0$$

$$|x-3| = 0$$

$$\Rightarrow x-3 = 0 \Rightarrow \boxed{x = 3} \quad .S.S = \{3\}$$

(8) $\sqrt{x^2} - 2x = 6$

notice that $\sqrt{x^2} = |x|$

Solution: $|x| - 2x = 6$

$$\Rightarrow |x| = 2x+6$$

$x \geq 0$

$$x = 2x+6$$

$$-x = 6$$

$$x = -6 \quad (\times)$$

$x < 0$

$$x = -2x-6$$

$$3x = -6$$

$$x = -2 < 0 \quad (\checkmark)$$

$$S.S = \{-2\}$$

$\boxed{9}$ $x^2 - 5|x| + 6 = 0$

$(x^2 = |x^2|)$

Solution

$$|x^2| - 5|x| + 6 = 0$$

$$(|x| - 2)(|x| - 3) = 0$$

$|x| = 2$ | $|x| = 3$

$x = \pm 2$ | $x = \pm 3$

$$S.S = \{2, -2, 3, -3\}$$

$\boxed{10}$ $\sqrt{x^2 - 4x + 4} = 4$

Solution:

$$\sqrt{(x-2)^2} = 4$$

$$\Rightarrow |x-2| = 4$$

$x \geq 2$ | $x < 2$

$x - 2 = 4$ | $x - 2 = -4$

$x = 6$ ✓ | $x = -2$ ✓

$$S.S = \{-2, 6\}$$

$\boxed{11}$ $5|3-x| - 2\sqrt{x^2 - 6x + 9} = 12$

$\Rightarrow 5|x-3| - 2\sqrt{(x-3)^2} = 12$

$\Rightarrow 5|x-3| - 2|x-3| = 12$

$\Rightarrow 3|x-3| = 12 \Rightarrow |x-3| = 4$

$x - 3 = 4$ | $x - 3 = -4$

$x = 7 > 3$ (✓) | $x = -1 < 3$ (✓)

$S.S = \{-1, 7\}$

12) $|x^2-1| = |x-1|$

Solution:

$$|(x-1)(x+1)| = |x-1|$$

$$\Rightarrow |x-1|\,|x+1| - |x-1| = 0$$

$$|x-1|\left(|x+1| - 1\right) = 0$$

$\lvert x-1\rvert = 0$	$\lvert x+1\rvert - 1 = 0$
$x-1 = 0$	$\lvert x+1\rvert = 1$
$x = 1$	

$x+1 = 1$	$x+1 = -1$
$x = 0$	$x = -2$

$$\Rightarrow S.S = \{1, 0, -2\}$$

13) $(x-5)^2 = |2x-10|$

Solution:

$$x^2 = |x^2|$$

$$|(x-5)^2| = 2|x-5|$$

$$|(x-5)^2| - 2|x-5| = 0$$

$$|x-5|\left(|x-5| - 2\right) = 0$$

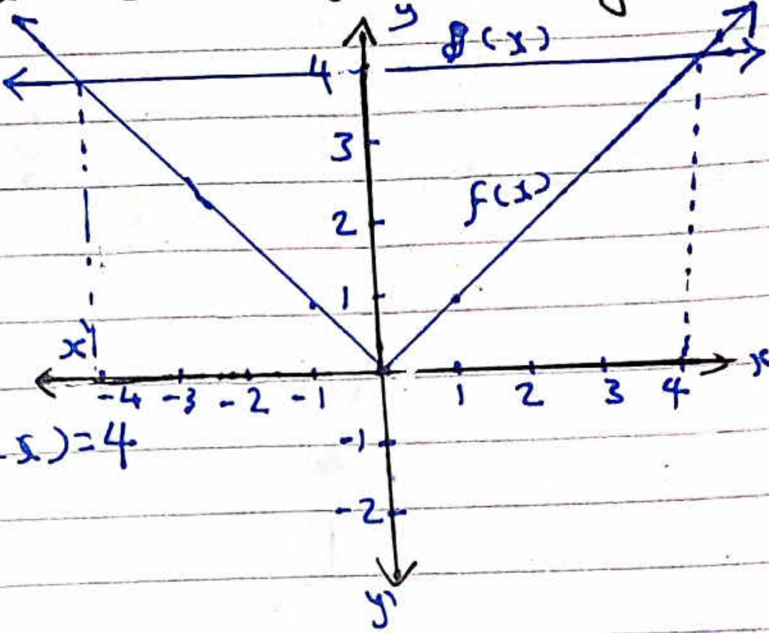
$\lvert x-5\rvert = 0$	$\lvert x-5\rvert - 2 = 0$
	$\lvert x-5\rvert = 2$
$x-5 = 0$	
$x = 5$	

$x-5 = 2$	$x-5 = -2$
$x = 7$	$x = 3$

$$S.S = \{5, 7, 3\}$$

Find graphically in $\mathbb{R}$ the solution set of each of the following equations and verify the result algebraically.

① $|x| - 4 = 0$



Solution

$|x| = 4$

first graphically

let $f(x) = |x|$, $g(x) = 4$

$S.S = \{4, -4\}$

algebraically: $|x| = 4 \Rightarrow x = \pm 4$

$S.S = \{4, -4\}$

② $|x| + 2 = 0$

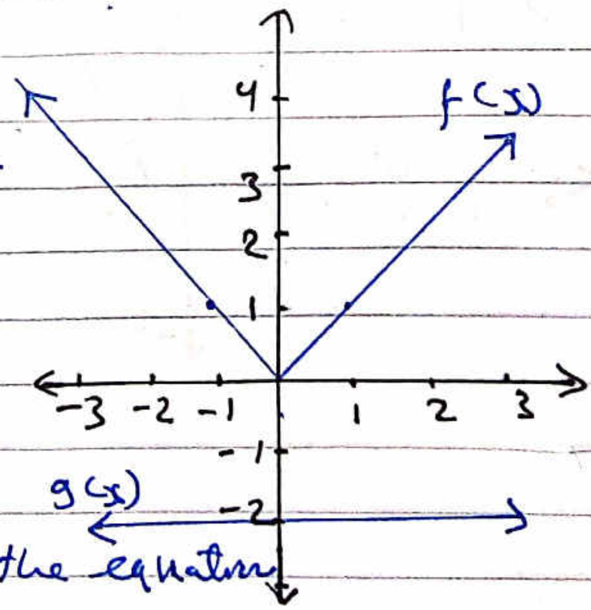
Solution

$|x| = -2$

let $f(x) = |x|$,

$g(x) = -2$

from the graph, the two curves don't intersect at any point $S.S = \phi$



algebraically:

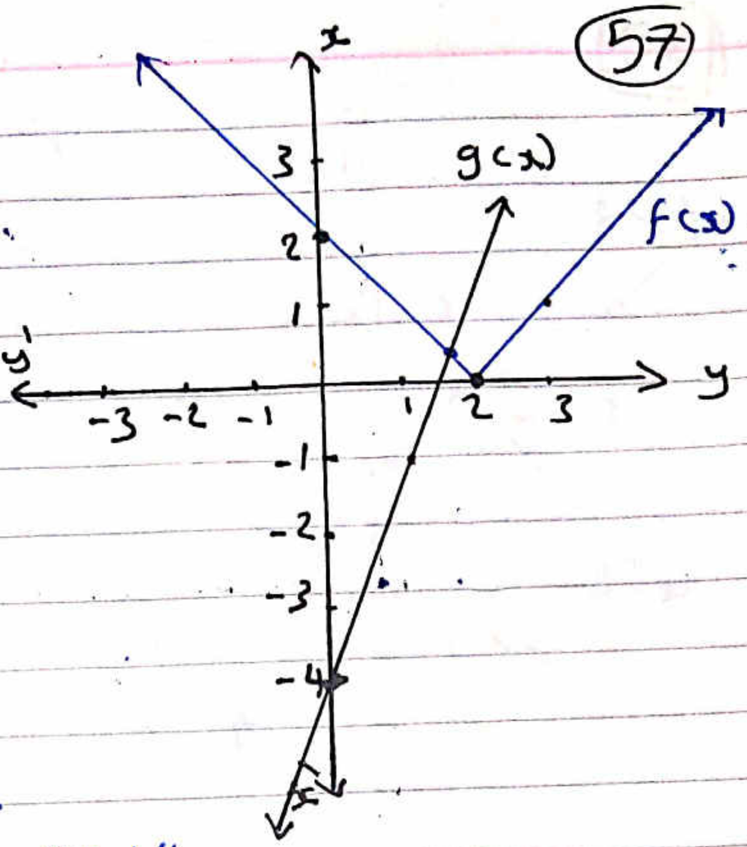
$|x| = -2$

there is no solution to the equation $|x| = -2$ in $\mathbb{R}$ $S.S = \phi$

③ $|x-2| = 3x-4$

Solution:

First graphically:



Let $f(x) = |x-2|$

$g(x) = 3x-4$

from the graph:

$S.S = \{1.5\}$

algebraically:

$|x-2| = 3x-4$

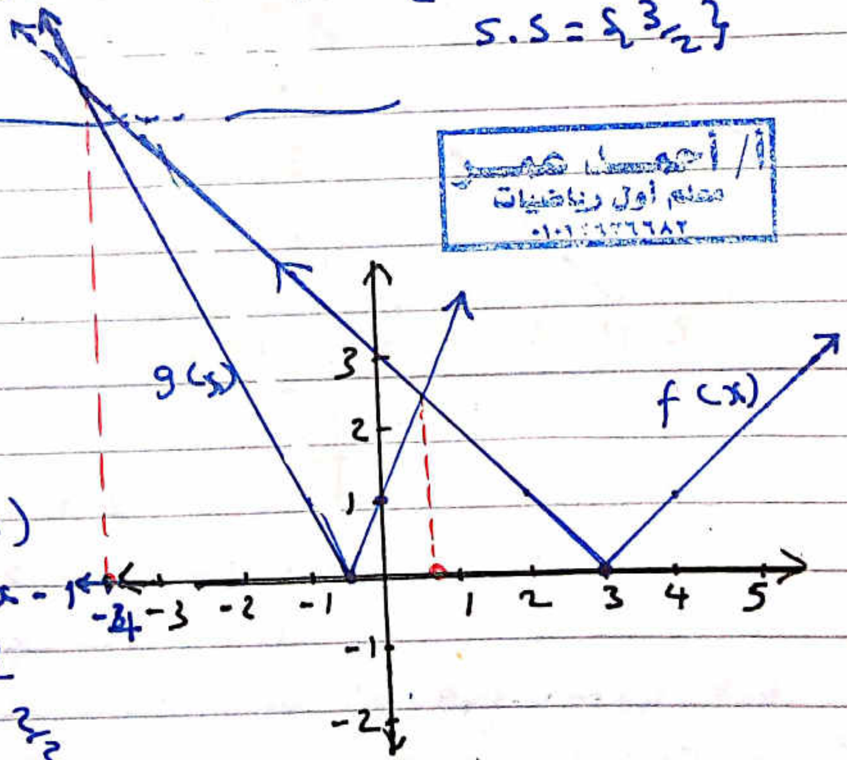
$x \geq 2$	$x < 2$
$x-2 = 3x-4$	$x-2 = -3x+4$
$-2x = -2$	$4x = 6 \quad \therefore x = 3/2$ (✓)
$x = -1$ (✗)	

$S.S = \{3/2\}$

④ $|x-3| = |2x+1|$

Let $f(x) = |x-3|$

$g(x) = |2x+1|$



algebraically

$x-3 = \pm(2x+1)$

$x-3 = 2x+1$	$x-3 = -2x-1$
$-x = 4$	$3x = 2$
$x = -4$	$x = 2/3$

$S.S = \{-4, 2/3\}$

⑤ $|x-2| + |x-1| = 0$

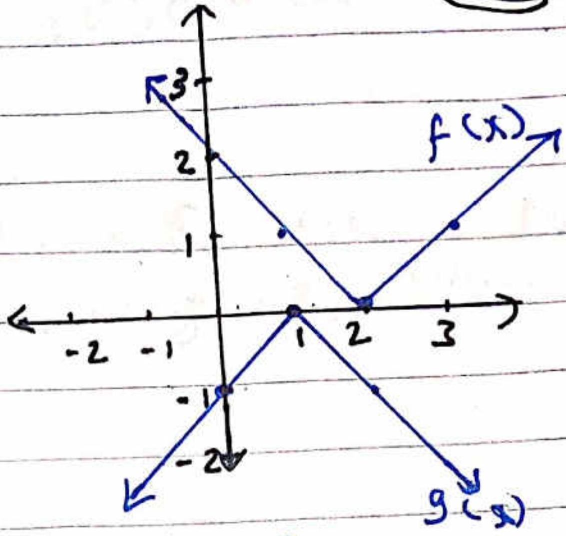
$\Rightarrow |x-2| = -|x-1|$

let $f(x) = |x-2|$

$g(x) = -|x-1|$

from the graph

$S.S = \phi$



algebraically:

$|x-2| = -|x-1|$

there is no solution to the equation

$|x-2| = -|x-1|$ in $\mathbb{R}$

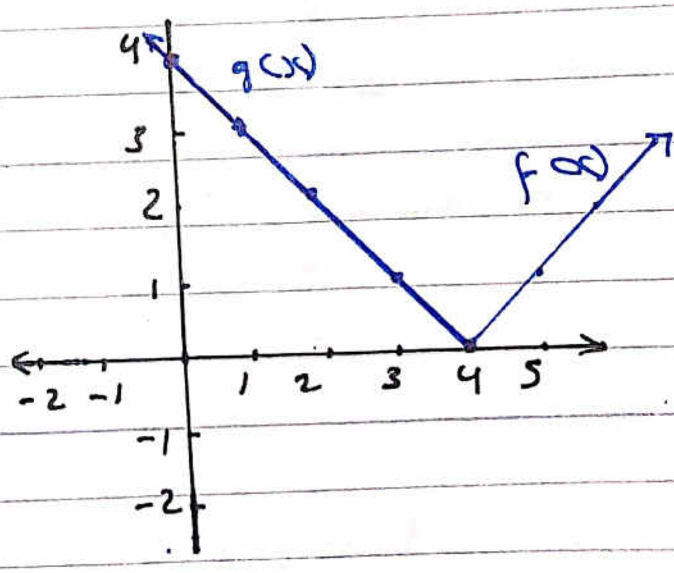
⑥ $|x-4| = 4-x$

let $f(x) = |x-4|$

$g(x) = 4-x$

$S.S =]-\infty, 4]$

algebraically:



$x-4 = \pm(4-x)$

$x \geq 4$	$x < 4$
$x-4 = 4-x$	$x-4 = -4+x$
$2x = 8$	$\Rightarrow -4 = -4$
$x = 4$	this relation is performed for all values of $x < 4$ $(x \in]-\infty, 4[)$

$S.S =]-\infty, 4]$

Second: Solving absolute value inequalities:

Example: Find algebraically in $\mathbb{R}$ the solution set of each of the following inequalities:

① $|x-3| \le 5$

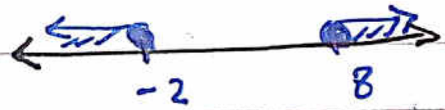
$$-5 \le x-3 \le 5$$
$$-2 \le x \le 8$$

$\therefore S.S = [-2, 8]$

② $|x-3| \ge 5$

$x - 3 \ge 5$ | $x - 3 \le -5$

$x \ge 8$ | $x \le -2$



$S.S = \mathbb{R} -]-2, 8[$

③ $|3x+2| + 5 < 4$

$$|3x+2| < 4-5$$
$$|3x+2| < -1$$

$\therefore S.S = \phi$

④ $\dfrac{1}{|3x|} \ge 5$

Solution:

$$|3x| \le \frac{1}{5}$$

$$-\frac{1}{5} \le 3x \le \frac{1}{5} \qquad \div 3$$

$$-\frac{1}{15} \le x \le \frac{1}{15}$$

$\therefore S.S = [-\frac{1}{15}, \frac{1}{15}] - \{0\}$

5. $$\frac{1}{|2x-3|} > 2$$

Solution:

$$|2x-3| < \frac{1}{2}$$

$\therefore$

$$-\frac{1}{2} < 2x-3 < \frac{1}{2}$$

$$\frac{5}{2} < 2x < \frac{7}{2} \qquad \div 2$$

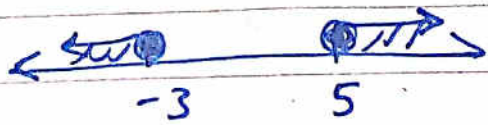
$$\frac{5}{4} < x < \frac{7}{4} \qquad \therefore S.S =]\frac{5}{4}, \frac{7}{4}[- \{\frac{3}{2}\}$$

6. $$\sqrt{x^2 - 2x + 1} \geq 4$$

Solution:

$$\sqrt{(x-1)^2} \geq 4$$

$$\Rightarrow |x-1| \geq 4$$

$x - 1 \geq 4$	$x - 1 \leq -4$
$x \geq 5$	$x \leq -3$
$S.S = \mathbb{R} -]-3, 5[$	

7. $$|x-2| + |2-x| < 6$$

$|x-2| = |2-x|$

Solution

$$|x-2| + |x-2| < 6$$

$$\Rightarrow 2|x-2| < 6 \qquad \Rightarrow |x-2| < 3$$

$$\Rightarrow -3 > x-2 < 3 \qquad \Rightarrow -1 > x > 5$$

$$S.S =]-1, 5[$$

8) $|3x-2| + 2|6x-4| \geq 20$

Solution

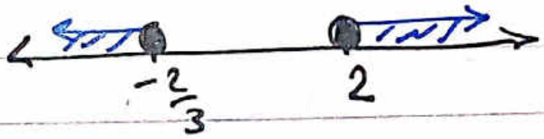
$$|3x-2| + 4|3x-2| \geq 20$$

$$\Rightarrow 5|3x-2| \geq 20$$

$$\Rightarrow |3x-2| \geq 4$$

	or
$\therefore 3x-2 \geq 4$	$3x-2 \leq -4$
$3x \geq 6$	$3x \leq -2$
$x \geq 2$	$x \leq -\frac{2}{3}$

$S.S = \mathbb{R} -]-\frac{2}{3}, 2[$



9) $\sqrt{(x+2)^2} + |2x+4| \geq 6$

Solution:

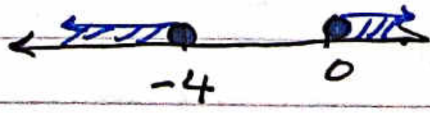
$$\therefore |x+2| + 2|x+2| \geq 6$$

$$\therefore 3|x+2| \geq 6 \qquad \div 3$$

$$|x+2| \geq 2$$

$x+2 \geq 2$	$x+2 \leq -2$
$x \geq 0$	$x \leq -4$

$S.S = \mathbb{R} -]-4, 0[$



Example 3: Find graphically in $\mathbb{R}$ S.S of each of the following inequalities, then verify the result algebraically:

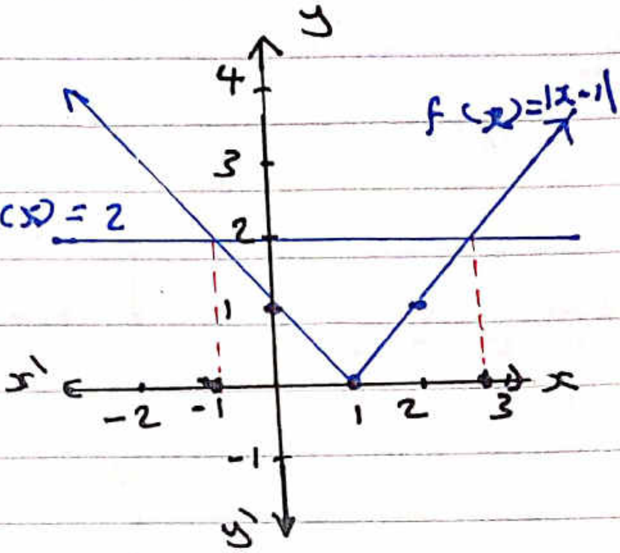
[1] $|x-1|<2$

Solution:

let $f(x)=|x-1|$

$g(x)=2$

the S.S. of $f(x)<g(x)$ in $\mathbb{R}$ is $]-1,3[$



algebraically:

$|x-1|<2 \Rightarrow -2<x-1<2$

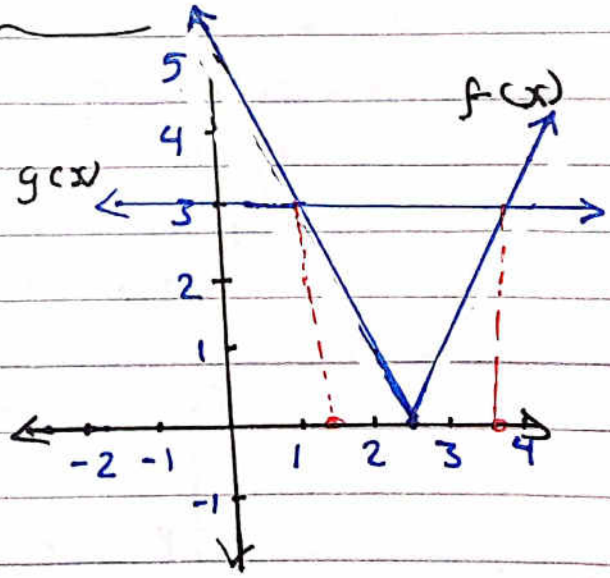
$\Rightarrow -1<x<3 \Rightarrow \therefore S.S=]-1,3[$

[2] $|2x-5|\geq 2$

Solution:

let $f(x)=|2x-5|$

, $g(x)=2$



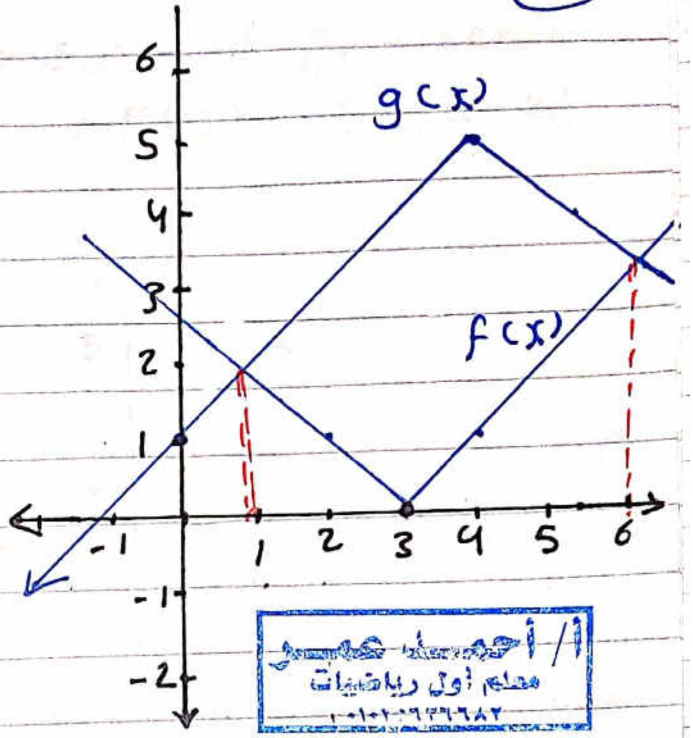
algebraically:

$2x-5\geq 2$	$2x-5\leq -2$
$2x\geq 7$	$2x\leq 3$
$x\geq \frac{7}{2}$	$x\leq \frac{3}{2}$

$S.S=\mathbb{R}-]\frac{3}{2},\frac{7}{2}[$

Scanned by CamScanner

③ $|x-3| + |x-4| > 5$



$|x-3| > 5 - |x-4|$

let $f(x) = |x-3|$

$g(x) = 5 - |x-4|$

from the graph

$S.S = \mathbb{R} - [1, 6]$

algebraically.

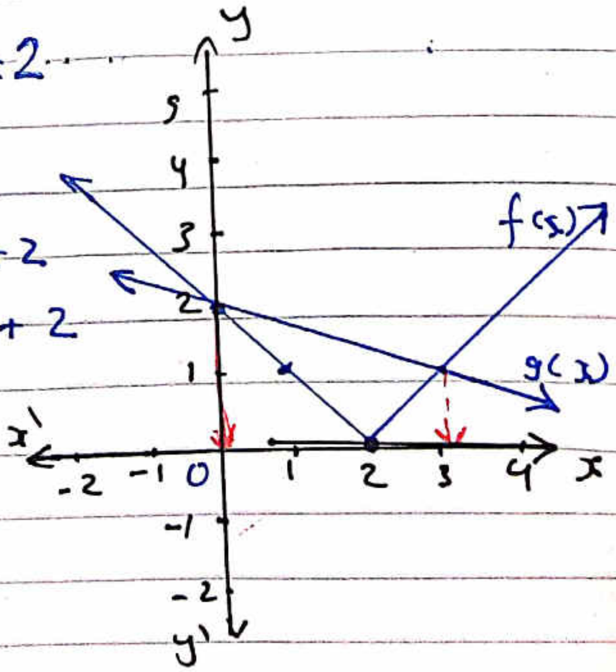
$|x-3| + |x-4| > 5$

$x > 4$	$x < 3$
$x - 3 + x - 4 > 5$	$x - 3 + x - 4 < -5$
$2x - 7 > 5$	$2x - 7 < -5$
$2x > 12$	$2x < 2$
$x > 6$	$x < 1$

$S.S = \mathbb{R} - [1, 6]$

④ $|x-2| < -\frac{1}{3}x + 2$

let $f(x) = |x-2|$, $g(x) = -\frac{1}{3}x + 2$



algebraically:

$\frac{1}{3}x - 2 < x - 2 < -\frac{1}{3}x + 2$

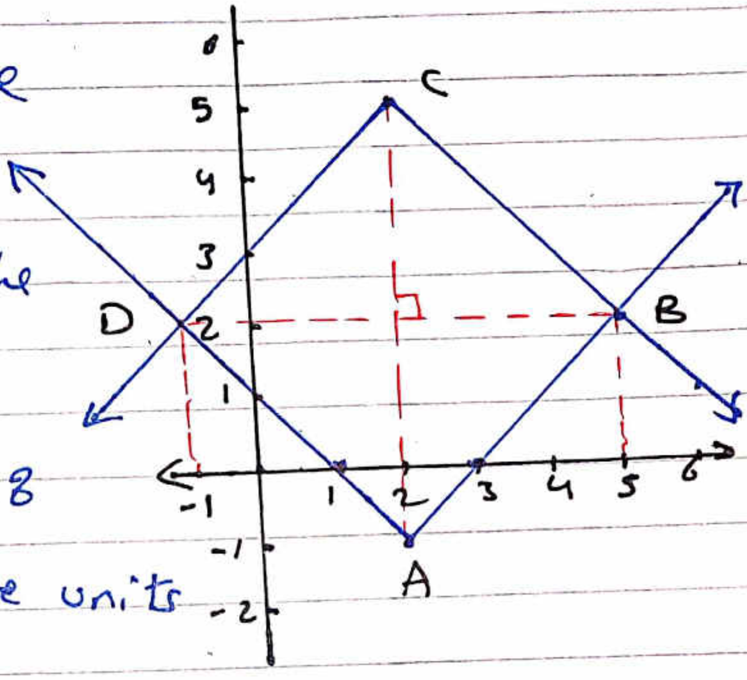
$\frac{1}{3}x - 2 < x - 2$	$x - 2 < -\frac{1}{3}x + 2$
$-\frac{2}{3}x < 0$	$\frac{4}{3}x < 4$
$x > 0$	$x < 3$

$S.S =]0, 3[$

Find in square units the area between the curves of the functions f and g where:

$f(x) = |x-2| - 1$, $g(x) = 5 - |x-2|$

from the graph ABCD is a square



The area between the two curves of the functions f and g

$= \frac{1}{2}(AC)^2 = \frac{1}{2} \times 6^2 = 18$ square units

Unit 2:

Lesson 1: Rational exponent

Definition:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

where $a \in \mathbb{R}^+$, $n \in \mathbb{Z}^+ - \{1\}$

or $a < 0$, n is an odd integer > 1

Notes

① $\sqrt[n]{a^n} = |a|$ if n is an even

② $\sqrt[n]{a^n} = a$ if n is an odd

③ If $a \in \mathbb{R}^-$: $a^{\frac{1}{n}} = \sqrt[n]{a} \in \mathbb{R}$ if n is an odd

, $a^{\frac{1}{n}} = \sqrt[n]{a} \notin \mathbb{R}$ if n is an even

④ $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ where

m, n haven't common factor , $n > 1$, $\sqrt[n]{a} \in \mathbb{R}$

⑤ If $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\sqrt[n]{a}$, $\sqrt[n]{b} \in \mathbb{R}$ then

$$\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b} \quad , \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \quad , \; b \neq 0$$

Find in $\mathbb{R}$ the solution set of each of the following equations

① $x^5 = 243$

Solution

$x = (243)^{\frac{1}{5}}$

$x = 3$

$S.S = \{3\}$

② $x^3 = -64$

Solution

$x = (-64)^{\frac{1}{3}}$

$x = -4$

$S.S = \{-4\}$

③ $x^4 = -16$

Solution:

$x = (-16)^{\frac{1}{4}}$

$\because -16 < 0$, 4 is even

$\therefore S.S = \phi$

④ $x^{\frac{7}{2}} = 128$

Solution

$\because 7$ is an odd

$\therefore x = (128)^{\frac{2}{7}}$

$\therefore x = 4$

$S.S = \{4\}$

⑤ $x^{\frac{5}{2}} = \frac{1}{32}$

Solution

$\because 5$ is an odd

$\therefore x = \left(\frac{1}{32}\right)^{\frac{2}{5}}$

$x = \frac{1}{4}$ $\quad \therefore S.S = \{\frac{1}{4}\}$

⑥ $\sqrt[3]{(x-1)^5} = 32$

Solution

$(x-1)^{5/3} = 32$

$x - 1 = (32)^{\frac{3}{5}} = 8$

$x = 9$

$S.S = \{9\}$

⑦ $(2x+3)^{\frac{4}{3}} = 81$

$\because 4$ is an even

$2x + 3 = \pm (81)^{3/4}$

$2x + 3 = \pm 27$

$2x + 3 = 27$	$2x + 3 = -27$
$2x = 24$	$2x = -30$
$x = 12$	$x = -15$

$S.S = \{12, -15\}$

8) $(x+1)^{-\frac{5}{2}} = (32)^{-\frac{1}{2}}$

Solution

$(x+1) = \left((32)^{-\frac{1}{2}}\right)^{-\frac{2}{5}}$

$x+1 = (32)^{\frac{1}{5}}$

$x+1 = 2$

$x = 1$

$S.S = \{1\}$

9) $(x^2-5x+9)^{\frac{5}{2}} = 243$

$x^2-5x+9 = (243)^{\frac{2}{5}}$

$x^2-5x+9 = 9$

$x^2-5x = 0$

$x(x-5) = 0$

$x=0,\ x=5$

$S.S = \{0, 5\}$

10) $x^{\frac{4}{5}} - 5x^{\frac{2}{5}} + 4 = 0$

$(x^{\frac{2}{5}} - 1)(x^{\frac{2}{5}} - 4) = 0$

$x^{\frac{2}{5}} = 1$	$x^{\frac{2}{5}} = 4$
$x = \pm 1$	$x = \pm 4^{\frac{5}{2}}$
	$x = \pm 32$

$S.S = \{-1, 1, 32, -32\}$

11) $x^{\frac{4}{3}} - 10x^{\frac{2}{3}} + 9 = 0$

Solution

$(x^{\frac{2}{3}} - 1)(x^{\frac{2}{3}} - 9) = 0$

$x^{\frac{2}{3}} = 1$	$x^{\frac{2}{3}} = 9$
$x = \pm 1$	$x = \pm 27$

$S.S = \{1, -1, 27, -27\}$

12) $x + 15 = 8\sqrt{x}$

$x - 8\sqrt{x} + 15 = 0$

$(\sqrt{x} - 3)(\sqrt{x} - 5) = 0$

$\sqrt{x} = 3$	$\sqrt{x} = 5$
$x = 9$	$x = 25$

$S.S = \{9, 25\}$

13) $\sqrt[5]{x^4} - 3\sqrt[5]{x^2} = 4$

$(x^{\frac{4}{5}} - 3x^{\frac{2}{5}} - 4 = 0$

$(x^{\frac{2}{5}} + 1)(x^{\frac{2}{5}} - 4) = 0$

$x^{\frac{2}{5}} = -1$	$x^{\frac{2}{5}} = 4$
$\because 2$ is even	$x = \pm(4)^{\frac{5}{2}}$
$\therefore x \notin R$	$x = \pm 32$

$S.S = \{32, -32\}$

Unit 2 : lesson 2
Exponential function

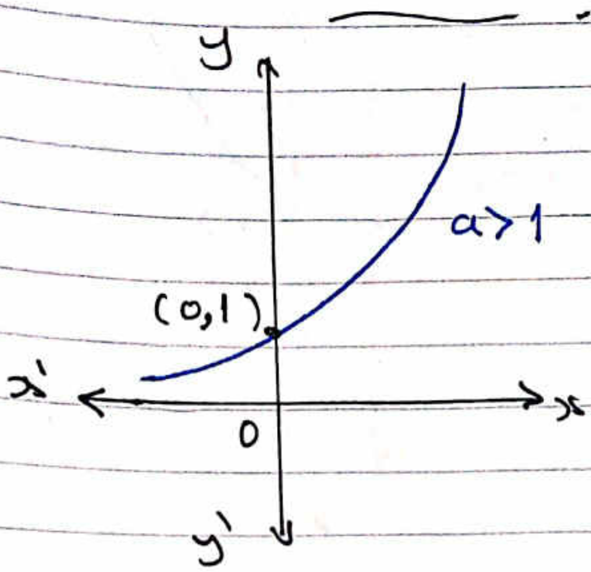
Definition:

If $a \in \mathbb{R}^+ - \{1\}$ then
$f : \mathbb{R} \rightarrow \mathbb{R}^+$ where $f(x) = a^x$
is called exponential function

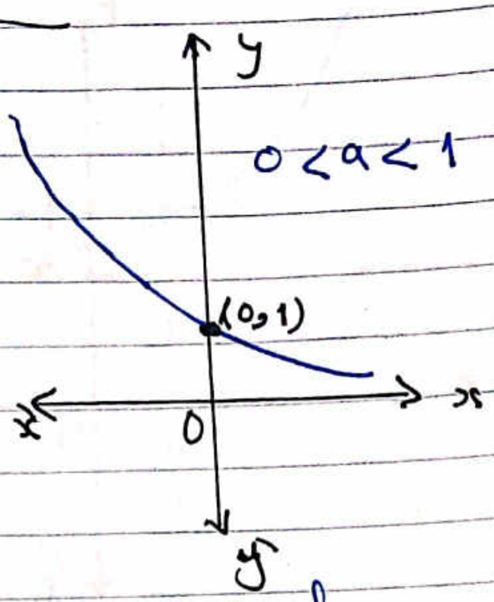
properties of : $f : f(x) = a^x$

* The domain $= \mathbb{R}$
* The range $= \mathbb{R}^+$
* The function is increasing on its domain $\mathbb{R}$ where $\boxed{a > 1}$ "exponential growth function"
* The function is decreasing on its domain $\mathbb{R}$ where $\boxed{0 < a < 1}$ "exponential decay function"
* The curve of the exponential function passes through the point $(0,1)$
* $f : f(x) = a^x$ is one-to-one

* The graphical representation of the exponential function:



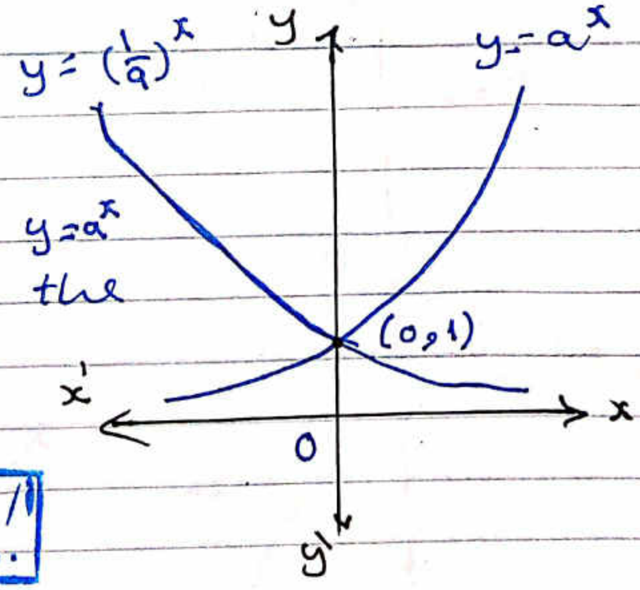
exponential growth function



exponential decay function

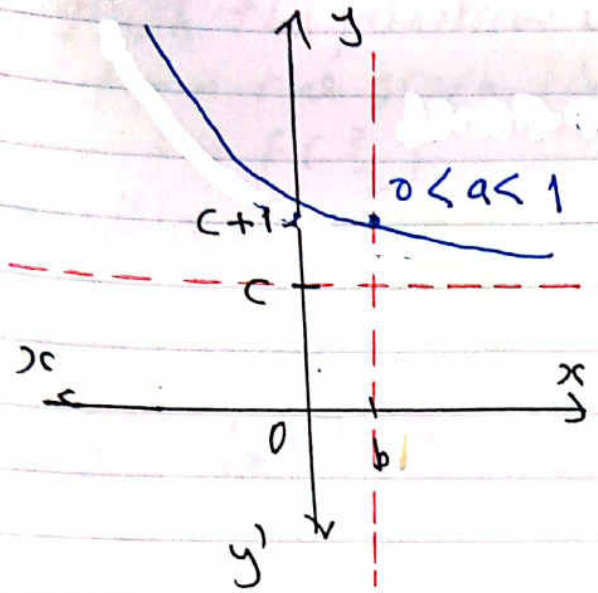
If $f(x) = a^x \rightarrow f(-x) = a^{-x} = \left(\frac{1}{a}\right)^x$

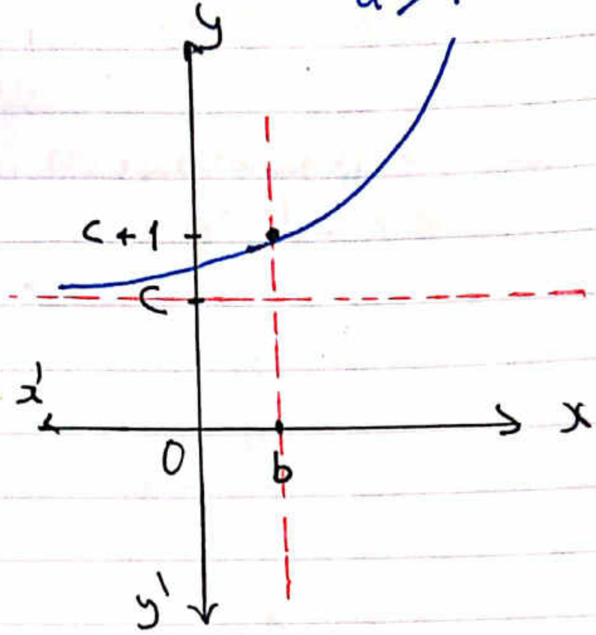
where $a > 1$



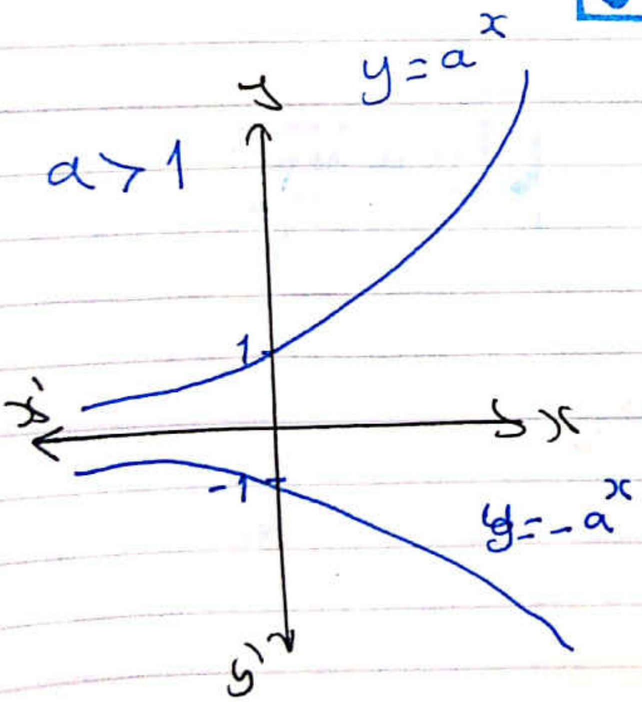
we notice that the curve of $y = a^x$ is the image of the curve of $y = \left(\frac{1}{a}\right)^x$

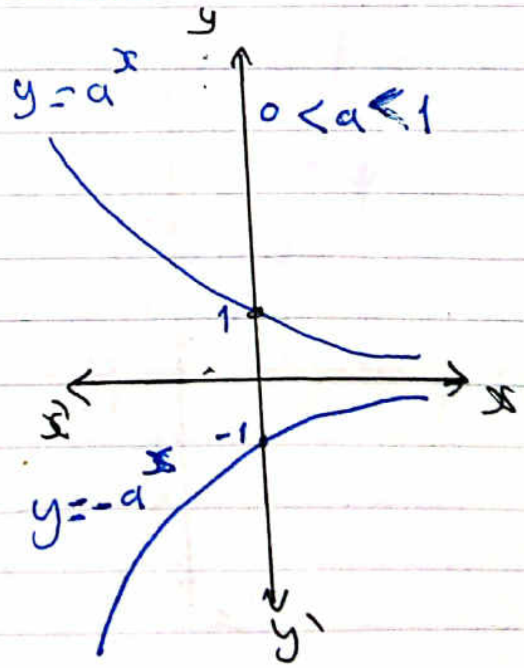
$f(x) = a^{x-b} + c$





The reflection of the exponential function in x-axis:





Example: If $f: \mathbb{R} \to \mathbb{R}^+$ where $f(x) = 3^{x-1}$

graph the function where $x \in [-2, 3]$,
from the graph find:

(1) $f(\frac{3}{2})$

(2) the value of x when: $3^{x-1} = 7\frac{1}{2}$

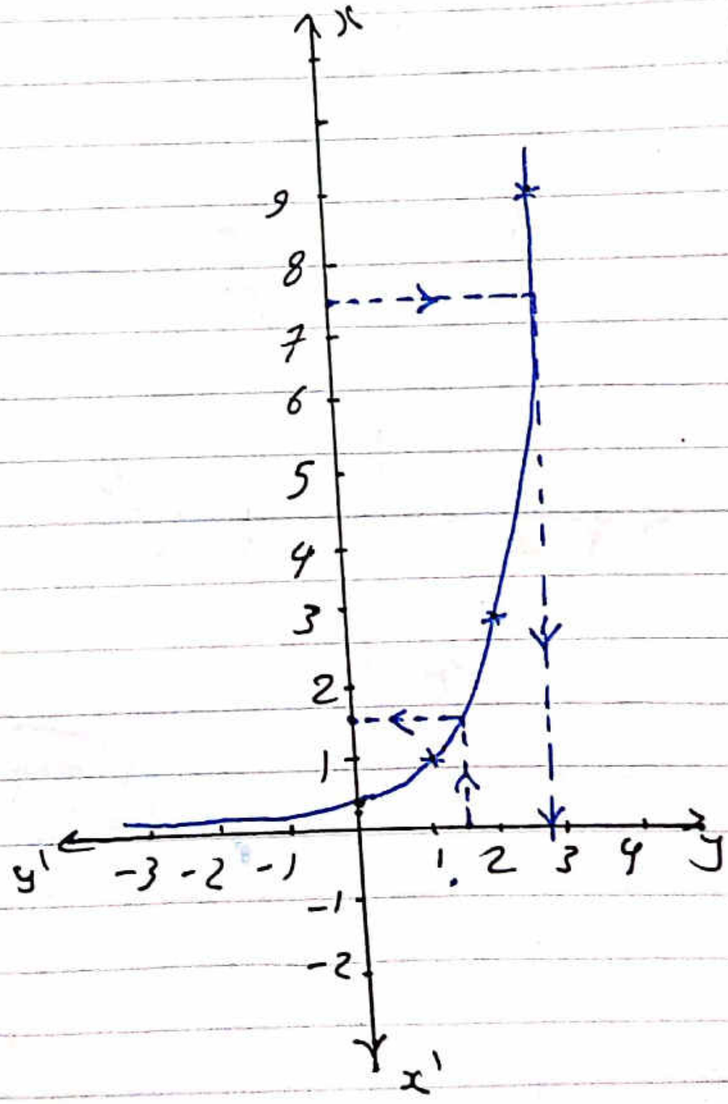
Solution

x	-2	-1	0	1	2	3
$f(x)$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9

(1) $f(\frac{3}{2}) \simeq 1.7$

(2) when $3^{x-1} = 7\frac{1}{2}$

$x \simeq 2.8$

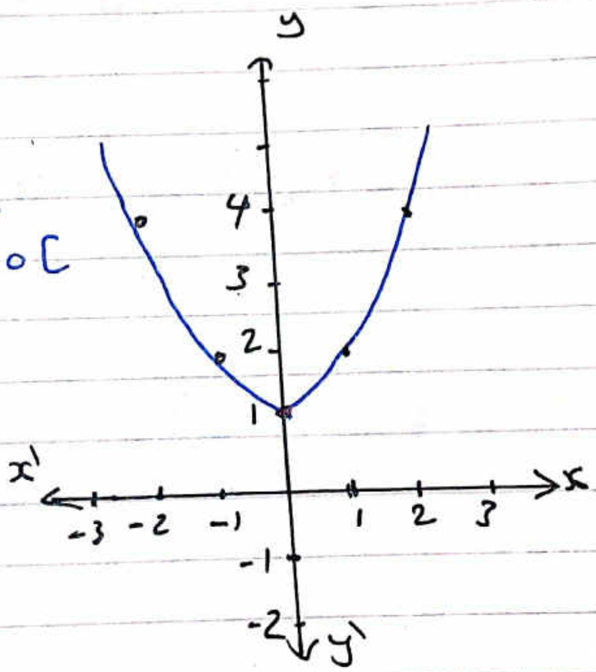


Example Draw the function $f: f(x) = 2^{|x|}$ then from the graph, deduce the range of the function and its monotonicity and show whether it is odd, even or otherwise. (72)

Solution: $f(x) = \begin{cases} 2^{x} & x \geq 0 \\ 2^{-x} & x < 0 \end{cases}$

$x < 0$			$x \geq 0$		
-2	-1	0	0	1	2
4	2	1	1	2	4

- the range $= [1, \infty[$
- increasing on $]0, \infty[$
- decreasing on $]-\infty, 0[$
- even



Applications on the exponential growth and decay

Saving: Ziad deposit L.E. 80000 in a bank which gives an annual interest of 10.5 %, find the total amount of money after 10 years.
Given that the total acount is given by

$C = a(1+r)^t$ where t is the number of years, a is the starting amount, r is the annual interest

Solution:

$a = 80000$, $r = 0.105$, $t = 10$

$C = a(1+r)^t$

$= 80000(1+0.105)^{10} = 217126$

Sports the number of spectators of a football team decreases at the rate of 4% each match as a result of recurrent loss in a championship. If the number of spectators in the first match was 36400 write the exponential function wich represents the number of spectators (y) in the match (t) then estimate the number of fans in the tenth match. (74)

Solution:

$$y = a(1-r)^t$$

where $a = 36400$, $r = \frac{4}{100}$, $t = 10$

$$y = 36400(1-0.04)^{10} \simeq 24200$$

A man deposited a capital of L.E. 5000 in one of the banks with annual compound interest 8 % Find the sum of the capital after 10 year in each of the following:
(1) the interest compounded annually.
(2) the interest compounded quarter annually
(3) the interest compounded monthly.

Solution:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$P = $ L.E. 5000 , $r = \frac{8}{100} = 0.08$, $t = 10$

(1) $n = 1$

$$A = 5000\left(1+\frac{0.08}{1}\right)^{10\times1} = \text{L.E. } 10794.6$$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٧٣٦٣٢١٩٨٢

(2) $n = 4$

$$A = 5000\left(1+\frac{0.08}{4}\right)^{10\times4} = \text{L.E. } 11040.2$$

(3) $n = 12$

$$A = 5000\left(1+\frac{0.08}{12}\right)^{10\times12} = \text{L.E. } 11098.2$$

Unit 2: lesson 3

Exponential equations

laws of exponents

for every $m, n \in \mathbb{Z}$ and $a, b \in \mathbb{R} - \{-1, 0, 1\}$

we have:

(1) If $a^n = 1$, then $n = \text{zero}$

(2) If $a^m = a^n$, then $m = n$

(3) If $a^n = b^n$,

then. $a = b$ when n is odd

and $a = \pm b$ when n is even

, if $a \neq b$, then $n = \text{zero}$

Find in $\mathbb{R}$ the solution set of each of the following equations (77)

① $2^{x+1} = 4$

$\because 2^{x+1} = 2^2$

$\therefore x+1 = 2$

$\therefore \boxed{x = 1}$

$\therefore S.S = \{1\}$

② $3^{x-1} = \frac{1}{9}$

$\therefore 3^{x-1} = 3^{-2}$

$\therefore x-1 = -2 \quad \therefore \boxed{x = -1}$

$S.S = \{-1\}$

③ $7^{x-2} = 1$

$\therefore 7^{x-2} = 7^0$

$\therefore x-2 = 0 \Rightarrow \boxed{x = 2}$

$S.S = \{2\}$

④ $5^{x+3} = 4^{x+3}$

$\because 5 \neq 4$

$\therefore x+3 = 0$

$\therefore x = -3$

$S.S = \{-3\}$

⑤ $x^{x^2-25} = 3^{x^2-25}$

Solution

$x = \pm 3$

or

$x^2 - 25 = 0$

$(x-5)(x+5) = 0$

$x = \pm 5$

$S.S = \{-3, 3, 5, -5\}$

⑥ $(3\sqrt{3})^{|x|} = 27$

$(3 \times 3^{\frac{1}{2}})^{|x|} = 3^3$

$(3^{3/2})^{|x|} = 3^3$

$3^{\frac{3}{2}|x|} = 3^3$

$\therefore \frac{3}{2}|x| = 3$

$|x| = 2$

$x = \pm 2$

$S.S = \{2, -2\}$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٣٦٣٦٦٨٢

Scanned by CamScanner

78

⑦ $3^{|3x-4|} = 9^{2x-2}$

$3^{|3x-4|} = (3^2)^{2x-2}$

$\Rightarrow 3^{|3x-4|} = 3^{4x-4}$

$\therefore |3x-4| = 4x-4$

$x \geq \frac{4}{3}$	$x < \frac{4}{3}$
$3x-4 = 4x-4$	$3x-4 = -4x+4$
$-x = 0$	$7x = 8$
$x = 0 \not\geq \frac{4}{3}$	$x = \frac{8}{7}$
Refused	

⑧ $\left(\frac{3}{5}\right)^{2x-1} = \frac{27}{125}$

$\left(\frac{3}{5}\right)^{2x-1} = \left(\frac{3}{5}\right)^3$

$2x-1 = 3 \Rightarrow \boxed{x = 2}$

$S.S = \{2\}$

⑨ $5^{x-1} \times 7^{1-x} = \frac{25}{49}$

$\Rightarrow 5^{x-1} \times \left(\frac{1}{7}\right)^{x-1} = \frac{25}{49}$

$\Rightarrow \left(\frac{5}{7}\right)^{x-1} = \left(\frac{5}{7}\right)^2$

$x-1 = 2 \Rightarrow \boxed{x = 3}$

$S.S = \{3\}$

⑩ $\sqrt{9^x - 2 \times 3^{x+1} + 9} = 24$

$\Rightarrow \sqrt{9^x - 6 \times 3^x + 9} = 24$

$\Rightarrow \sqrt{(3^x - 3)^2} = 24$

$\Rightarrow |3^x - 3| = 24$

$x \geq 1$	$x < 1$
$3^x - 3 = 24$	$3^x - 3 = -24$
$3^x = 27$	$3^x = -21$
$3^x = 3^3$	refused
$\boxed{x = 3}$	

⑪ $9^{x^2-1} = \frac{1}{27^x}$

$9^{x^2-1} = 27^{-x}$

$3^{2(x^2-1)} = 3^{-3x}$

$\therefore 2x^2 - 2 = -3x$

$\Rightarrow 2x^2 + 3x - 2 = 0$

$(2x-1)(x+2) = 0$

$\therefore x = \frac{1}{2}, \ x = -2$

$S.S = \{\frac{1}{2}, -2\}$

79

Find in $\mathbb{R}$ the solution set of each of the following equations:

① $3^{x+3} - 3^{x+2} = 162$

Solution:

$3^{x+2}(3-1) = 162$

$3^{x+2} \times 2 = 162$

$3^{x+2} = 81$

$3^{x+2} = 3^4$

$x+2 = 4$

$\boxed{x = 2}$

$S.S = \{2\}$

② $5^{2x} + 25 = 26 \times 5^x$

$\Rightarrow 5^{2x} - 26 \times 5^x + 25 = 0$

$(5^x - 1)(5^x - 25) = 0$

$5^x = 1$ | $5^x = 25$

$\therefore \boxed{x = 0}$ | $5^x = 5^2$

$\therefore \boxed{x = 2}$

$S.S = \{0, 2\}$

③ $2^x + 2^{5-x} = 12$

multiplying two sides by 2^x

$2^{2x} + 2^5 = 12 \times 2^x$

$2^{2x} - 12 \times 2^x + 32 = 0$

$(2^x - 4)(2^x - 8) = 0$

$2^x = 2^2$ | $2^x = 2^3$

$\boxed{x = 2}$ | $\boxed{x = 3}$

$S.S = \{2, 3\}$

④ $\left(\frac{1}{2}\right)^{x+1} + \left(\frac{1}{2}\right)^{x+3} + \left(\frac{1}{2}\right)^{x+5} = 84$

$\left(\frac{1}{2}\right)^{x+1}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4\right] = 84$

$\left(\frac{1}{2}\right)^{x+1} \times \frac{21}{16} = 84$

$\left(\frac{1}{2}\right)^{x+1} = 64$

$\left(\frac{1}{2}\right)^{x+1} = 2^6 = \left(\frac{1}{2}\right)^{-6}$

$\therefore x+1 = -6$

$x = -7$

$S.S = \{-7\}$

Find in $\mathbb{R}$ the solution set of each of the following equations: (80)

(1) $3^{x^2-42} = \left(\frac{1}{3}\right)^x$ ② $7^{2-x} + 7^{-x} = 50$

Solution:

[1] $3^{x^2-42} = 3^{-x}$

$\therefore x^2 - 42 = -x$

$x^2 + x - 42 = 0$

$(x-6)(x+7) = 0$

$x = 6 , x = -7$

$S.S = \{6, -7\}$

[2] $7^{-x}(7^2+1) = 50$

$7^{-x} \times 50 = 50$

$7^{-x} = 1$

$-x = 0$

$x = 0$

$S.S = \{0\}$

[3] $3^x \times 5^y = 75$, $3^y \times 5^x = 45$

Solution: $3^x \times 5^y = 75$ ...① , $3^y \times 5^x = 45$ ...②

Dividing (1) by (2)

$\left(\frac{3}{5}\right)^x \times \left(\frac{3}{5}\right)^{-y} = \frac{75}{45} = \frac{5}{3} \Rightarrow \left(\frac{3}{5}\right)^{x-y} = \left(\frac{3}{5}\right)^{-1}$

$\Rightarrow \boxed{x - y = -1}$ ... ③

multiplying (1) by (2)

$(15)^x \times (15)^y = 75 \times 45 \Rightarrow (15)^{x+y} = (15)^3$

$\Rightarrow \boxed{x + y = 3}$ ... ④ from 3, 4

$x = 1 , y = 2$ $S.S = \{(1, 2)\}$

Scanned by CamScanner

If $f(x) = 7^{x+1}$, then find the value of x that satisfies:

$$f(2x-1) + f(x-2) = 50$$

Solution:

$\because f(x) = 7^{x+1}$

$\therefore f(2x-1) = 7^{2x-1+1} = 7^{2x}$

, $f(x-2) = 7^{x-2+1} = 7^{x-1}$

$\therefore 7^{2x} + 7^{x-1} = 50 \qquad \times 7$

$7 \times 7^{2x} + 7^{x} - 350 = 0$

$(7 \times 7^{x} + 50)(7^{x} - 7) = 0$

$7^{x} = -\frac{50}{7}$ (Refused) | $7^{x} = 7$, $\boxed{x = 1}$

Example: If $f(x) = 2^{x}$ prove that $\frac{f(x+1)}{f(x-1)} + \frac{f(x-1)}{f(x+1)} = \frac{17}{4}$

Solution

$$L.H.S = \frac{2^{x+1}}{2^{x-1}} + \frac{2^{x-1}}{2^{x+1}} = 2^{x+1-x+1} + 2^{x-1-x-1}$$

$$= 2^{2} + 2^{-2}$$

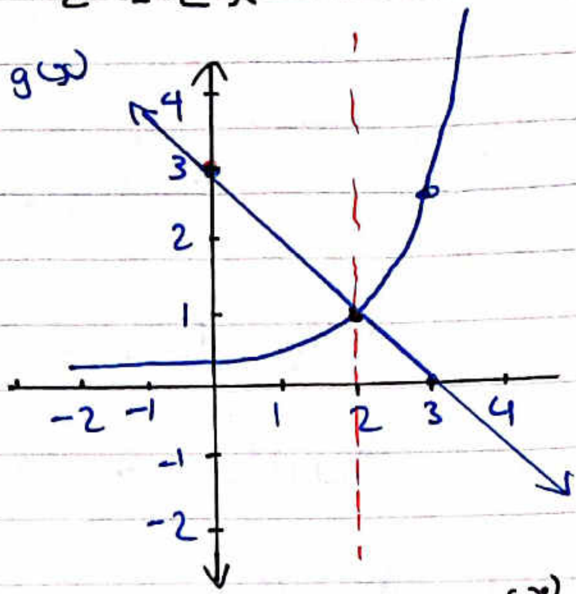
$$= 4 + \frac{1}{4} = \frac{17}{4}$$

$$= R.H.S$$

Scanned by CamScanner

Find graphically in $\mathbb{R}$ the solution set of each of the following equations (82)

(1) $3^{x-2} = 3 - x$ (2) $2^{x} = 2x$



Solution

① let $f(x) = 3^{x-2}$

, $g(x) = 3 - x$

$S.S = \{2\}$

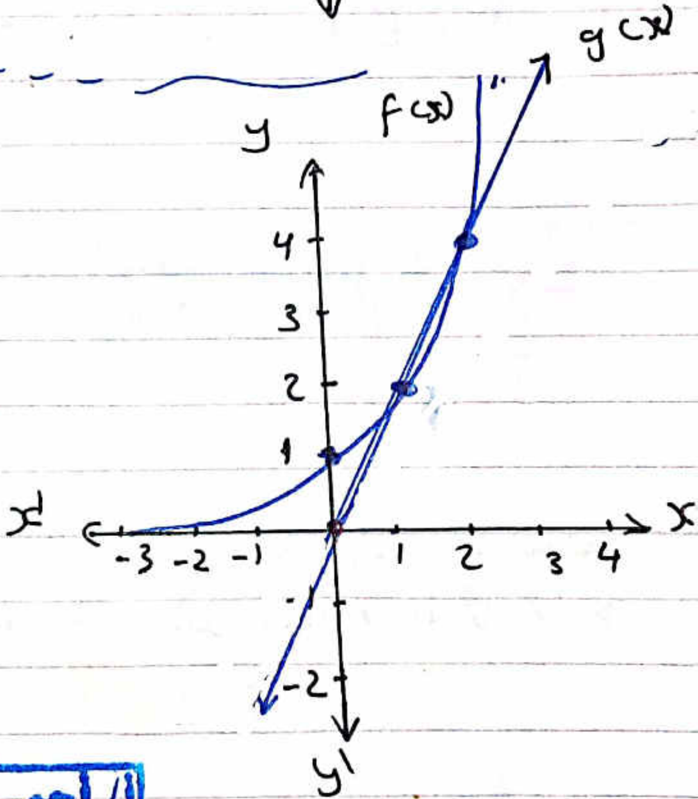
(2)

let $f(x) = 2^{x}$

$g(x) = 2x$

from the graph

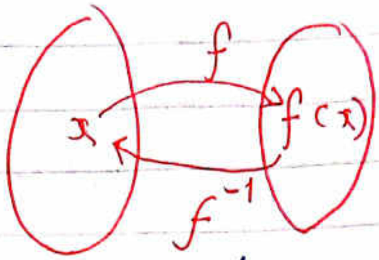
$S.S = \{1, 2\}$



Unit 2:

Lesson 4: The inverse function

* If f is one-to-one function then, for each $(x, y) \in f$, we find $(y, x) \in f^{-1}$



* We can find the rule of f^{-1} directly by replacing the two variable x, y, then finding y in terms of x

Notes:

① $f^{-1}(x) \neq \frac{1}{f(x)}$

② each of the function f and its inverse f^{-1} are symmetric about the straight line $\boxed{y = x}$

③ The two functions f, g are said to be each of them is the inverse function of the other if $(f \circ g)(x) = (g \circ f)(x) = x$

④ the domain of f = the range of f^{-1}
the range of f = the domain of f^{-1}

Example: Find the inverse function for each of the following:

1) $f = \{(1,2), (2,3), (3,4)\}$

$f^{-1} = \{(2,1), (3,2), (4,3)\}$

2)

x	-2	1	2	5
$f(x)$	7	4	1	-1

Solution:

x	7	4	1	-1
$f^{-1}(x)$	-2	1	2	5

3) $f(x) = \frac{1}{2}x + 4$

Solution:

$y = \frac{1}{2}x + 4$ putting $x = \frac{1}{2}y + 4$

$\Rightarrow \frac{1}{2}y = x - 4 \Rightarrow y = 2(x - 4)$

$\Rightarrow f^{-1}(x) = 2x - 8$

4) $f(x) = 4x$

Solution: $y = 4x \Rightarrow$ putting $x = 4y$

$\Rightarrow y = \frac{1}{4}x \Rightarrow f^{-1}(x) = \frac{1}{4}x$

5. $f(x) = 5 + \frac{4}{x}$

Solution: putting $x = 5 + \frac{4}{y}$

$$\Rightarrow \frac{4}{y} = x - 5 \Rightarrow \frac{y}{4} = \frac{1}{x-5}$$

$$\Rightarrow y = \frac{4}{x-5}$$

$$f^{-1}(x) = \frac{4}{x-5}$$

6. $f(x) = 8x^3 - 1$

Solution: $y = 8x^3 - 1$

replacing x, y

$$x = 8y^3 - 1 \Rightarrow 8y^3 = x + 1 \Rightarrow y^3 = \frac{x+1}{8}$$

$$\Rightarrow y = \sqrt[3]{\frac{x+1}{8}} \Rightarrow f^{-1}(x) = \frac{1}{2}\sqrt[3]{x+1}$$

7. $f(x) = \sqrt[3]{4-x}$

Solution: $y = \sqrt[3]{4-x}$ replacing x, y

$$x = \sqrt[3]{4-y} \Rightarrow x^3 = 4 - y \Rightarrow y = 4 - x^3$$

$$f^{-1}(x) = 4 - x^3$$

Scanned by CamScanner

Example: for each of the following functions Find ① The domain and the range of the function f

② $f^{-1}(x)$ and determine the range and the domain of f^{-1}

[1] $f(x) = 2 + \sqrt{3-x}$

Solution $\because y = 2 + \sqrt{3-x} \Rightarrow x \leq 3$

the domain $=]-\infty, 3]$

$\therefore y \geq 2$ the range $= [2, \infty[$

putting $x = 2 + \sqrt{3-y} \Rightarrow \sqrt{3-y} = x - 2$

$\Rightarrow 3 - y = (x-2)^2 \Rightarrow y = 3 - (x-2)^2$

for every $x \geq 2$, $y \leq 3$

$\therefore$ the domain of $f^{-1} = [2, \infty[$

, the range of $f^{-1} =]-\infty, 3]$

② $f(x) = x^2$, $x \geq 0$

Solution $y = x^2$ the domain $= [0, \infty[$

putting $x = y^2 \Rightarrow y^2 = x$

$\because x \geq 0 \therefore y = \sqrt{x}$ $\therefore y \geq 0$

the domain of $f^{-1} = [0, \infty[$

the range of $f^{-1} = [0, \infty[$

(3) $f(x) = (x-1)^2 + 2$ where $x \geqslant 1$

Solution: $y = (x-1)^2 + 2$

$\because x \geqslant 1 \quad \therefore y \geqslant 2$

$\therefore$ the domain $= [1, \infty[$, the range $= [2, \infty[$

replacing the two variables x, y

$\therefore x = (y-1)^2 + 2 \Rightarrow (y-1)^2 = x - 2$

$\because x \geqslant 2$

$\therefore y - 1 = \sqrt{x-2} \Rightarrow y = 1 + \sqrt{x-2}$

$f^{-1}(x) = 1 + \sqrt{x-2}$

the domain of $f^{-1} = [2, \infty[$,

the range of $f^{-1} = [1, \infty[$

(4) $f(x) = x^2 + 8x + 7$ where $x \geqslant -4$

Solution

$y = x^2 + 8x + 7 = (x+4)^2 - 9$

$$y = \left(x - \left(-\frac{b}{2a}\right)\right)^2 + f\left(-\frac{b}{2a}\right)$$

$\because x \geqslant -4 \quad \therefore y \geqslant -9$

$\therefore$ the domain $= [-4, \infty[$

the range $= [-9, \infty[$

replacing the two variables x, y

$$x = (y+4)^2 - 9 \quad , x \geq -9 , y \geq -4$$

$$\Rightarrow (y+4)^2 = x+9$$

$$\therefore y+4 = \sqrt{x+9} \Rightarrow y = \sqrt{x+9} - 4$$

$$f^{-1}(x) = \sqrt{x+9} - 4$$

the domain of $f^{-1} = [-9, \infty[$
the range of $f^{-1} = [-4, \infty[$

(5) $f(x) = \sqrt{9-x^2}$ where $-3 \leq x \leq 0$

Solution:

$$y = \sqrt{9-x^2}$$

$\because -3 \leq x \leq 0 \quad \therefore 0 \leq y \leq 3$

the domain of $f = [-3, 0]$
the range of $f = [0, 3]$
replacing the two variables x, y

$$x = \sqrt{9-y^2} \Rightarrow \text{where } -3 \leq y \leq 0$$

$$\Rightarrow x^2 = 9 - y^2 \Rightarrow y^2 = 9 - x^2$$

$$\therefore y = -\sqrt{9-x^2} \quad \therefore f^{-1}(x) = -\sqrt{9-x^2}$$

the range of $f^{-1} = [-3, 0]$
the domain of $f^{-1} = [0, 3]$

6) $f(x) = \sqrt{9 - x^2}$ where $0 \le x \le 3$

Solution:

$y = \sqrt{9 - x^2}$ $\because 0 \le x \le 3$ $\therefore 0 \le y \le 3$

the domain $= [0, 3]$, the range $= [0, 3]$

replacing x, y

$\therefore x = \sqrt{9 - y^2}$, $0 \le y \le 3$

$\Rightarrow x^2 = 9 - y^2 \Rightarrow y^2 = 9 - x^2$

$\therefore y = \sqrt{9 - x^2} = f^{-1}(x) = \sqrt{9 - x^2}$

the domain of $f^{-1} = [0, 3]$

the range of $f^{-1} = [0, 3]$

90) Determine whether each of the two functions f, g is inverse function to the other or not in each of the following:

1) $f(x) = 2x - 3$, $g(x) = \frac{x+3}{2}$

Solution:

$$(f \circ g)(x) = f(g(x)) = \cancel{2}\left(\frac{x+3}{\cancel{2}}\right) - 3$$

$$= x + 3 - 3 = x$$

$$, (g \circ f)(x) = g(f(x)) = \frac{2x - 3 + 3}{2} = \frac{2x}{2} = x$$

$\therefore$ Each of f, g is the inverse function of the other.

2) $f(x) = \frac{-2}{x-5}$, $g(x) = \frac{5x-2}{x}$

Solution:

$$(f \circ g)(x) = f(g(x)) = \frac{-2}{\frac{5x-2}{x} - 5}$$

$$= \frac{-2}{\frac{5x - 2 - 5x}{x}} = \frac{-2x}{-2} = x$$

$$(g \circ f)(x) = g(f(x)) = \frac{5\frac{-2}{x-5} - 2}{\frac{-2}{x-5}}$$

$$= \frac{\left(5\frac{-2}{x-5} - 2\right)(x-5)}{-2} = \frac{-10 - 2x + 10}{-2}$$

$$= \frac{-2x}{-2} = x$$

Each of f, g is inverse function of the other

Example which of the following functions its inverse is the function itself:

1. $f(x) = 2x$

$y = 2x \Rightarrow x = 2y \Rightarrow y = \frac{1}{2}x \neq f(x)$

$\therefore f^{-1}(x) \neq f(x)$

2. $f(x) = \frac{1}{x-k} + k$ where $k \in R$

Solution $y = \frac{1}{x-k} + k$

replacing x, y $\therefore$ $x = \frac{1}{y-k} + k$

$\frac{1}{y-k} = x - k$

$\Rightarrow y - k = \frac{1}{x-k} \Rightarrow y = \frac{1}{x-k} + k$

$\therefore f^{-1}(x) = f(x)$

3. $f(x) = \frac{1}{x-3} + 5$

Solution:

$y = \frac{1}{x-3} + 5$ replacing x, y

$x = \frac{1}{y-3} + 5 \Rightarrow x - 5 = \frac{1}{y-3} \Rightarrow y - 3 = \frac{1}{x-5}$

$\Rightarrow y = \frac{1}{x-5} + 3 \neq f(x)$

$\therefore f^{-1}(x) \neq f(x)$

Logarithmic function and its graph

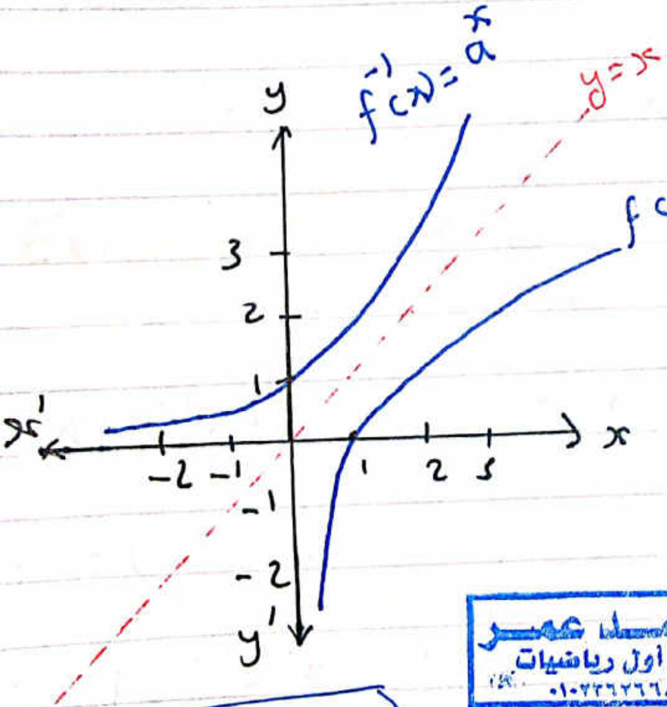
If $a \in \mathbb{R}^+ - \{1\}$ then the function

$f: \mathbb{R}^+ \longrightarrow \mathbb{R}$ where

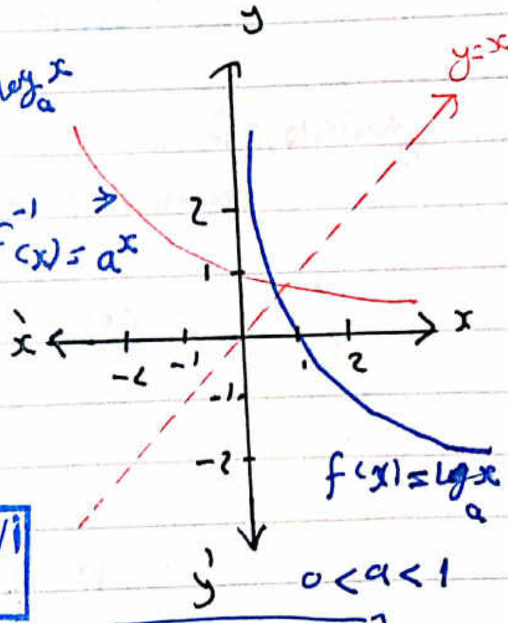
$$f(x) = \log_a x$$

is called the logarithmic function

* the domain $= \mathbb{R}^+$, the range $= \mathbb{R}$



أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٢٦٣٦٦٨٢



$\boxed{a > 1}$ $\boxed{0 < a < 1}$

increasing decreasing

* The logarithmic function is the inverse function for the exponential function

$$y = \log_a x \iff x = a^y$$

where $a \in \mathbb{R}^+ - \{1\}$, $x \in \mathbb{R}^+$, $y = \mathbb{R}$

Example: Express the following in the equivalent exponential form

$$\log_2 4\sqrt{2} = \frac{5}{2}$$

Solution:

$$\log_2 4\sqrt{2} = \frac{5}{2} \iff 4\sqrt{2} = 2^{\frac{5}{2}}$$

Example 2: Express each of the following in the logarithmic form:

① $(\sqrt{2})^4 = 4$ ② $5^0 = 1$ ③ $5^{-3} = \frac{1}{125}$

Solution:

① $(\sqrt{2})^4 = 4 \iff \log_{\sqrt{2}} 4 = 4$

② $5^0 = 1 \iff \log_5 1 = 0$

③ $5^{-3} = \frac{1}{125} \iff \log_5 \frac{1}{125} = -3$

Find the value of the following: 94

(1) $\log_2 16$ (2) $\log_8 1$ (3) $\log 0.0001$

(4) $\log_{1/2} 128$ (5) $\log_2 \frac{1}{8}$ (6) $\log_3 \sqrt[4]{27}$

Solution:

(1) let $x = \log_2 16$

$\Rightarrow 2^x = 16$

$\Rightarrow 2^x = 2^4$

$\therefore \boxed{x = 4} \Rightarrow \log_2 16 = 4$

(2) $\log_8 1 = x$

$\Rightarrow 8^x = 1$

$\therefore x = 0$

$\therefore \log_8 1 = 0$

(3) $\log 0.0001 = x$

$\Rightarrow 10^x = \frac{1}{10000}$

$\Rightarrow 10^x = 10^{-4}$

$\therefore x = -4$

$\therefore \log 0.0001 = -4$

(4) $\log_{1/2} 128 = x$

$\therefore (\frac{1}{2})^x = 128 = 2^7$

$\therefore (\frac{1}{2})^x = (\frac{1}{2})^{-7}$

$\therefore x = -7 \Rightarrow \log_{1/2} 128 = -7$

(5) $\log_2 \frac{1}{8} = x$

$\Rightarrow 2^x = \frac{1}{8} = \frac{1}{2^3}$

$\Rightarrow 2^x = 2^{-3} \Rightarrow x = -3$

$\therefore \log_2 \frac{1}{8} = -3$

(6) $\log_3 \sqrt[4]{27} = x$

$\Rightarrow 3^x = (27)^{\frac{1}{4}} = (3^3)^{\frac{1}{4}} = 3^{\frac{3}{4}}$

$\therefore x = \frac{3}{4}$

$\therefore \log_3 \sqrt[4]{27} = \frac{3}{4}$

Solve each of the following equations in $\mathbb{R}$: (95)

① $\log_{81} x = \frac{3}{4}$

Solution: $x = 81^{\frac{3}{4}}$

$\Rightarrow x = (3)^3 = 27$

② $\log_3 (2x-5) = 0$

$\Rightarrow 2x - 5 = 3^0$

$\Rightarrow 2x - 5 = 1$

$2x = 6$

$\boxed{x = 3}$ (verify)

③ $\log_2 (\log_3 x) = 1$

$\Rightarrow \log_3 x = 2^1 = 2$

$\Rightarrow x = 3^2 = 9$

④ $\log_5 |2x+1| = 1$

$\Rightarrow |2x+1| = 5^1 = 5$

$2x+1=5$	$2x+1=-5$
$2x=4$	$2x=-6$

$\boxed{x=2}$ or $\boxed{x=-3}$

⑤ $(\log_3 x)^2 - 9\log_3 x + 20 = 0$

$(\log_3 x - 4)(\log_3 x - 5) = 0$

$\log_3 x = 4$	$\log_3 x = 5$
$x = 3^4 = 81$	$x = 3^5 = 243$

⑥ $3^{\log_4 (x+1.25)} = \frac{1}{3}$

$\Rightarrow 3^{\log_4 (x+1.25)} = 3^{-1}$

$\Rightarrow \log_4 (x+1.25) = -1$

$\Rightarrow x + 1.25 = 4^{-1}$

$\Rightarrow x = \frac{1}{4} - 1.25 = -1$

⑦ $\log_2 (3^x - 3^{x-2}) = 3$

$\Rightarrow 3^x - 3^{x-2} = 2^3$

$\Rightarrow 3^{x-2}(3^2 - 1) = 8$

$\Rightarrow 3^{x-2} \times 8 = 8$

$\Rightarrow 3^{x-2} = 1$

$x - 2 = 0 \Rightarrow \boxed{x = 2}$

Determine the domain of each of the functions that are defined by the following rule:

① $f(x) = \log_3 (2x+1)$

$2x+1 > 0$

$\Rightarrow x > -\frac{1}{2}$

the domain $=]-\frac{1}{2}, \infty[$

② $f(x) = 2 \log x$

$x > 0$

$\therefore$ the domain $=]0, \infty[$

③ $f(x) = \log_{2-x} x$

$$\begin{cases} x > 0 \\ 2-x > 0 \\ 2-x \neq 1 \end{cases} \Rightarrow \begin{cases} x > 0 \\ x < 2 \\ x \neq 1 \end{cases}$$

the domain $=]0, 2[- \{1\}$

④ $f(x) = \log_{(5-x)} (x-3)$

$$\begin{cases} x-3 > 0 \\ 5-x > 0 \\ 5-x \neq 1 \end{cases} \Rightarrow \begin{cases} x > 3 \\ x < 5 \\ x \neq 4 \end{cases}$$

$\therefore$ The domain $=]3, 5[- \{4\}$

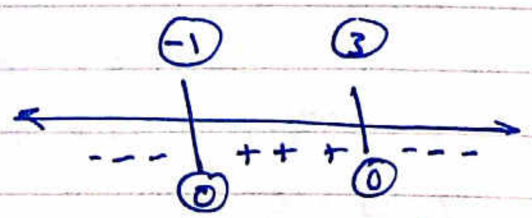
⑤ $f(x) = \log_{x-3} x^2$

$$\begin{cases} x^2 > 0 \\ x-3 > 0 \\ x-3 \neq 1 \end{cases} \Rightarrow \begin{cases} x^2 > 0 \\ x > 3 \\ x \neq 4 \end{cases}$$

$\therefore$ The domain $=]3, \infty[- \{4\}$

⑥ $f(x) = \log_4 (3-x)(x+1)$

$(3-x)(x+1) > 0$



$x \in]-1, 3[$ $\therefore$ The domain $=]-1, 3[$

Find in $\mathbb{R}$ the S.S of each of the following 97

① $\log_x 5x = 2$

$\Rightarrow 5x = x^2$

$\Rightarrow x^2 - 5x = 0$

$x(x-5) = 0$

$x = 0$ (refused)

$x = 5$ (verify)

$S.S = \{5\}$

② $\log_{-x} 81 = 4$

$\Rightarrow (-x)^4 = 81$

$\Rightarrow x^4 = 3^4$

$\Rightarrow x = 3$ (refused)

or $x = -3$ (verify)

$S.S = \{-3\}$

③ $\log_x (x^2 - 12) = 1$

$x^2 - 12 = x^1$

$x^2 - x - 12 = 0 \Rightarrow (x+3)(x-4) = 0$

$\Rightarrow x = -3$ (refused) or $x = 4$ (verify)

$S.S = \{4\}$

④ $\log_x (\sqrt{x-2} + 2) = 1$

Solution:

$\sqrt{x-2} + 2 = x^1$

$\Rightarrow \sqrt{x-2} = x - 2$

$\sqrt{x-2} - (\sqrt{x-2})^2$

$\Rightarrow \sqrt{x-2}(1 - \sqrt{x-2}) = 0$

$\Rightarrow x = 2$

$\sqrt{x-2} = 1$

$x - 2 = 1$

$x = 3$

$\therefore S.S = \{2, 3\}$

(98)

Use the curve of the function $f: f(x) = \log_2 x$ to represent each of the functions that are defined by the following rules, from the graph determine the domain, range and monotonicity of each of function (1) $l(x) = \log_2 (x-2)$

(2) $t(x) = \log_2 x + 1$ (3) $h(x) = -\log_2 x$ (4) $g(x) = \log_2 (-x)$

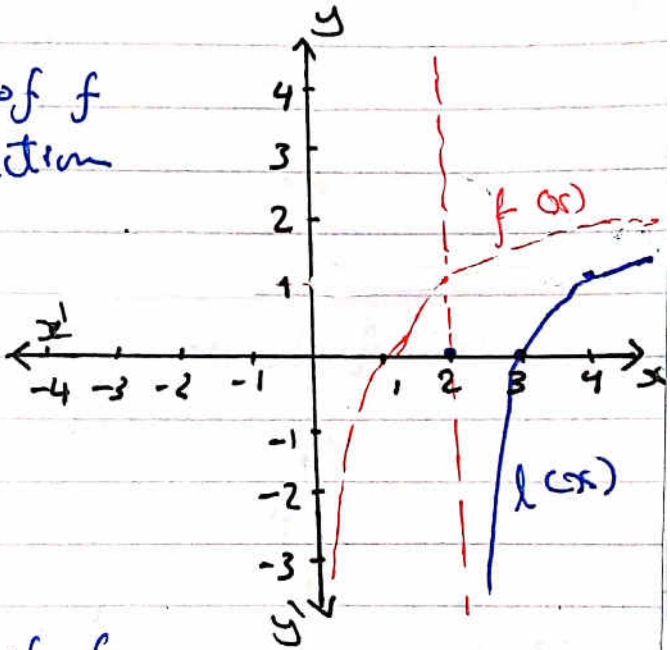
Solution:

[1] horizontal translation of f 2 units in $\overrightarrow{ox}$ direction

* domain $=]2, \infty[$

* range $= \mathbb{R}$

* Increasing on $]2, \infty[$

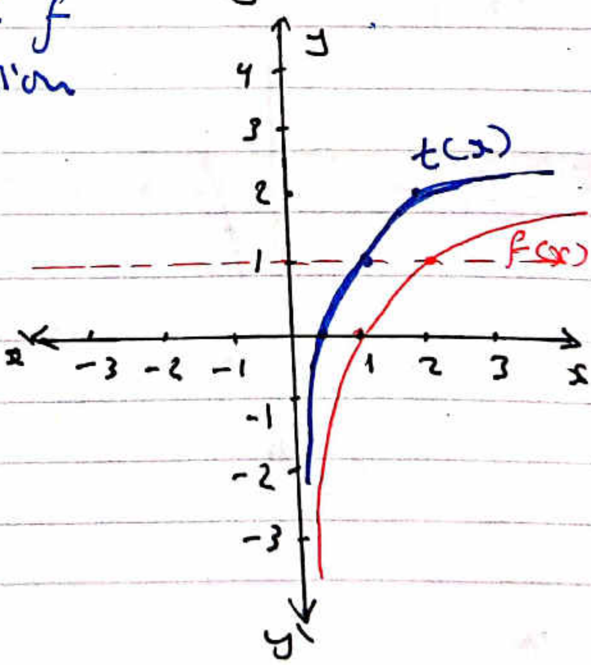


[2] vertical translation of f 1 unit in $\overrightarrow{oy}$ direction

* domain $=]0, \infty[$

* Range $= \mathbb{R}$

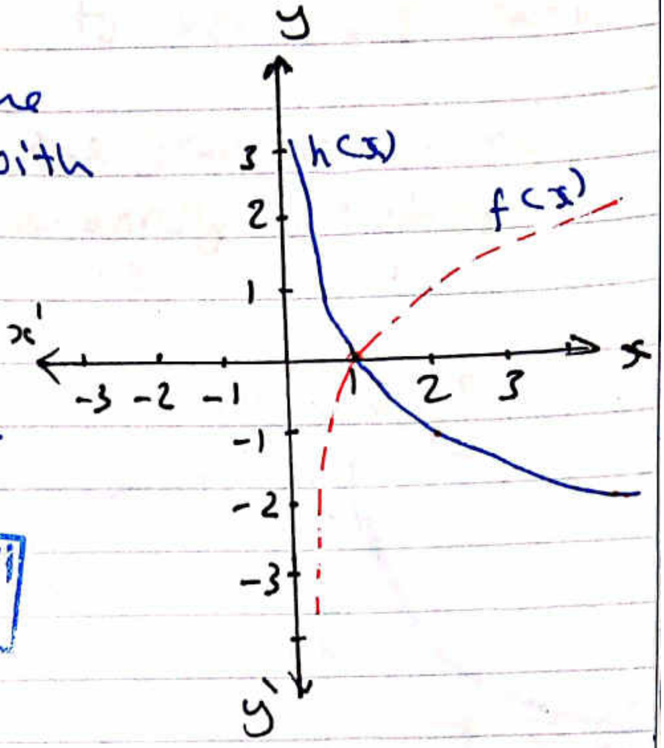
* increasing on $]0, \infty[$



③ $h(x) = -\log_2 x$

The curve of $h(x)$ is the same curve of $f(x)$ with reflection in x-axis

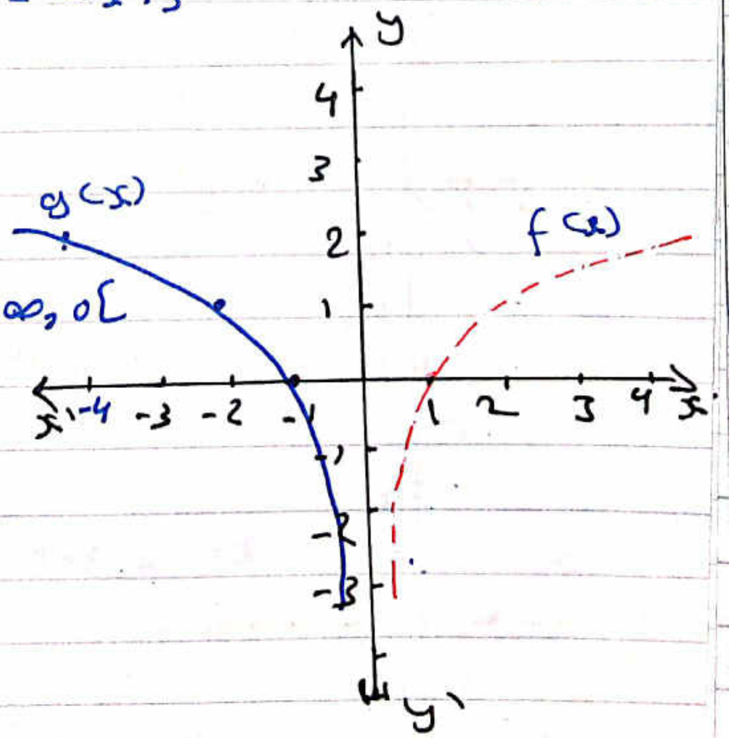
* domain = $]0, \infty[$
* Range = $\mathbb{R}$
* decreasing on $]0, \infty[$



④ $g(x) = \log_2(-x)$

the curve of $g(x)$ is the same of $f(x)$. with reflection in y-axis

* domain = $]-\infty, 0[$
* range = $\mathbb{R}$
* decreasing on $]-\infty, 0[$

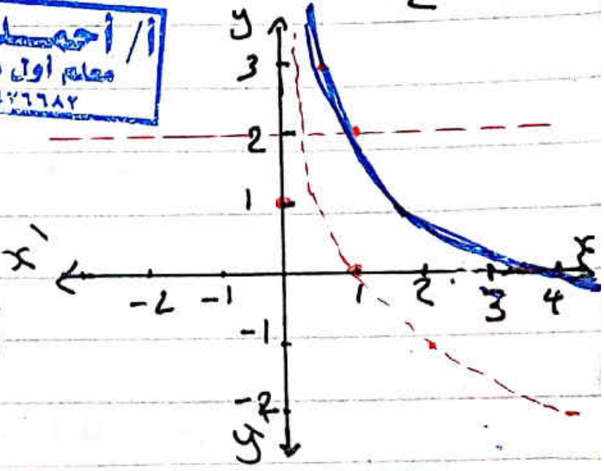


Example: Use the curve of the function $f: f(x) = \log_{\frac{1}{2}} x$ to represent each of the functions, from the graph determine domain, range and monotonicity of each of functions ① $g(x) = \log_{\frac{1}{2}} x + 2$

② $\ell(x) = \log_{\frac{1}{2}}(x+1)$ ③ $h(x) = -\log_{\frac{1}{2}} x$

Solution

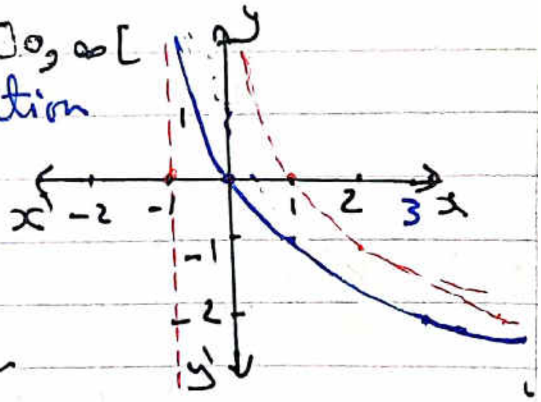
[1] The curve of $g(x)$ is the same curve of f with vertical translation 2 units in the direction $\overrightarrow{oy}$



* The domain $=]0, \infty[$
* The range $= \mathbb{R}$

The function is decreasing on $]0, \infty[$

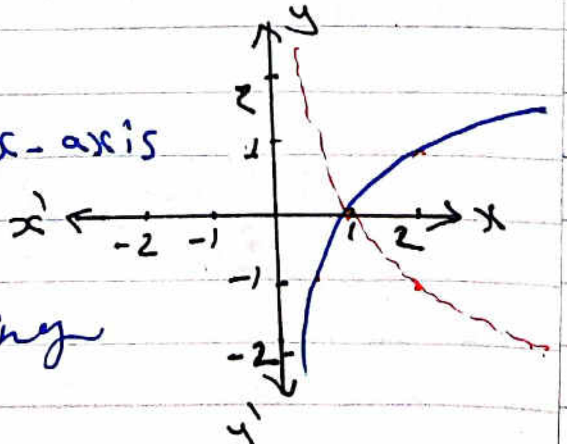
[2] ℓ with horizontal translation 1 unit in $\overrightarrow{ox}$ direction



* The range $=]-1, \infty[$
* the domain $= \mathbb{R}$

the function is decreasing on its domain $]-1, \infty[$

[3] with reflection in the x-axis



* The domain $=]0, \infty[$
the range $= \mathbb{R}$

the function is increasing on its domain

Unit 2: Lesson 6:

Some properties of logarithms

1) $\log_a a = 1$ where $a \in \mathbb{R}^+ - \{1\}$

2) $\log_a 1 = 0$ where $a \in \mathbb{R}^+ - \{1\}$

3) $\log_a xy = \log_a x + \log_a y$

where $x, y \in \mathbb{R}^+$, $a \in \mathbb{R}^+ - \{1\}$

4) $\log_a \frac{x}{y} = \log_a x - \log_a y$

where $x, y \in \mathbb{R}^+$, $a \in \mathbb{R}^+ - \{1\}$

Corollary

$$\log_a \frac{xy}{zl} = \log_a x + \log_a y - \log_a z - \log_a l$$

5) $\log_a x^n = n \log_a x$ $\quad x \in \mathbb{R}^+$, $n \in \mathbb{R}$, $a \in \mathbb{R}^+ - \{1\}$

6) $\log_y x = \frac{\log_a x}{\log_a y}$

7) $\log_y x = \frac{1}{\log_x y}$

Example: Without using calculator, find the value of each of the following

1) $\log 2 + \log 5 = \log 2 \times 5 = \log 10 = 1$

2) $\log_5 15 - \log_5 3 = \log_5 \frac{15}{3} = \log_5 5 = 1$

3) $\log_2 5 \times \log_5 2 = \log_2 5 \times \frac{1}{\log_2 5} = 1$

4) $\log 54 - 3\log 3 - \log 2 = \log 54 - \log 3^3 - \log 2$

$= \log \frac{54}{27 \times 2} = \log 1 = 0$

5) $\log 0.009 - \log \frac{27}{16} + \log 15\frac{5}{8} - \log \frac{1}{12}$

$= \log \frac{9}{1000} - \log \frac{27}{16} + \log \frac{125}{8} - \log \frac{1}{12}$

$= \log \dfrac{\frac{9}{1000} \times \frac{125}{8}}{\frac{27}{16} \times \frac{1}{12}} = \log 1 = 0$

6) $2\log 25 + \log(\frac{1}{3} + \frac{1}{5}) + 2\log 3 - \log 30$

$= \log 25^2 + \log \frac{8}{15} + \log 3^2 - \log 30$

$= \log \dfrac{625 \times \frac{8}{15} \times 9}{30} = \log 100 = \log 10^2 = 2\log 10$

$= 2$

(7) $1 + \log 3 - \log 2 - \log 15$

$= \log 10 + \log 3 - \log 2 - \log 15$

$= \log \dfrac{10 \times 3}{2 \times 15} = \log \dfrac{30}{30} = \log 1 = 0$

(8) $\log 25 + \dfrac{\log 8 \times \log 16}{\log 64}$

$= \log 25 + \dfrac{\log 2^3 \times \log 2^4}{\log 2^6}$

$= \log 25 + \dfrac{3 \log 2 \times 4 \cancel{\log 2}}{6 \cancel{\log 2}}$

$= \log 25 + 2 \log 2 = \log 25 + \log 2^2$

$= \log 25 \times 4 = \log 100 = 2$

(9) $\dfrac{1}{\log_2 12} + \dfrac{1}{\log_8 12} + \dfrac{1}{\log_9 12}$

$= \log_{12} 2 + \log_{12} 8 + \log_{12} 9 = \log_{12} 2 \times 8 \times 9 =$

$= \log_{12} 144 = \log_{12} 12^2 = 2$

(10) $\dfrac{\log 729 - \log 64}{\log 9 - \log 4} = \dfrac{\log \frac{729}{64}}{\log \frac{9}{4}} = \dfrac{\log (\frac{9}{4})^3}{\log (\frac{9}{4})} = \dfrac{3 \log \frac{9}{4}}{\log \frac{9}{4}}$

$= 3$

Find in $\mathbb{R}$ Solution set of each of the following: (104)

① $\log (x+6) = 2\log_3 x$

Solution:

$\log_3 (x+6) = \log_3 x^2 \Rightarrow x^2 = x+6$

$\Rightarrow x^2 - x - 6 = 0 \Rightarrow (x+2)(x-3) = 0$

$\Rightarrow x = -2$ (refused) $x = 3 \Rightarrow S.S = \{3\}$

② $\log (x+1) + \log (x-1) = \log (x+5)$

Solution:

$\log (x+1)(x-1) = \log (x+5)$

$\Rightarrow x^2 - 1 = x+5 \Rightarrow x^2 - x - 6 = 0$

$\Rightarrow x = -2$ (refused) $, x = 3 \Rightarrow S.S = \{3\}$

③ $\log (x+8) - \log (x-1) = 1$

$\log \dfrac{x+8}{x-1} = 1 \Rightarrow \dfrac{x+8}{x-1} = 10$

$\Rightarrow x+8 = 10x - 10 \Rightarrow 9x = 18 \Rightarrow x = 2 \Rightarrow S.S = \{2\}$

④ $\log_4 x = 1 - \log_4 (x-3)$

$\Rightarrow \log_4 x + \log_4 (x-3) = 1 \Rightarrow \log_4 x(x-3) = 1$

$\Rightarrow x(x-3) = 4 \Rightarrow (x^2 - 3x - 4 = 0 \Rightarrow (x-4)(x+1) = 0$

$\Rightarrow x = 4 \quad , \quad x = -1$ (refused) $\Rightarrow S.S = \{4\}$

⑤ $\log(8-x) + 2\log\sqrt{x-6} = 0$

$\log(8-x) + \log(\sqrt{x-6})^2 = 0$

$\Rightarrow \log(8-x)(x-6) = 0 \Rightarrow (8-x)(x-6) = 1$

$\Rightarrow 8x - 48 - x^2 + 6x - 1 = 0 \Rightarrow x^2 - 14x - 49 = 0$

$\Rightarrow (x-7)^2 = 0 \Rightarrow x = 7 \Rightarrow S.S = \{7\}$

⑥ $3^{\log x} = 2^{\log 3}$ taking logarithms of the two sides

$\therefore \log 3^{\log x} = \log 2^{\log 3}$

$\Rightarrow \log x \times \cancel{\log 3} = \cancel{\log 3} \times \log 2$

$\Rightarrow \log x = \log 2 \Rightarrow x = 2 \therefore S.S = \{2\}$

⑦ $x^{\log x} = 10$

$\log x^{\log x} = \log 10 \Rightarrow \log x \times \log x = 1$

$(\log x)^2 = 1 \Rightarrow \log x = \pm 1$

$\log x = 1 \Rightarrow x = 10$

$\log x = -1 \Rightarrow \log \frac{1}{x} = 1 \Rightarrow \frac{1}{x} = 10 \Rightarrow x = \frac{1}{10}$

$S.S = \{10, \frac{1}{10}\}$

(8) $\log_3 x = \log_x 3$

$\Rightarrow \log_3 x = \frac{1}{\log_3 x} \Rightarrow (\log_3 x)^2 = 1$

$\Rightarrow \log_3 x = \pm 1$

$\log_3 x = 1 \Rightarrow x = 3$, $\log_3 x = -1 \Rightarrow x = \frac{1}{3}$

$S.S = \{3, \frac{1}{3}\}$

(9) $(\log x)^2 - \log x^2 = 3$

$\Rightarrow (\log x)^2 - 2\log x - 3 = 0$

$(\log x + 1)(\log x - 3) = 0$

$\log x = -1 \Rightarrow x = \frac{1}{10}$, $\log x = 3 \Rightarrow x = 10^3 = 1000$

$S.S = \{\frac{1}{10}, 1000\}$

(10) $\log_2 x + \log_x 2 = 2$

$\log_2 x + \frac{1}{\log_2 x} = 2 \Rightarrow (\log_2 x)^2 - 2\log_2 x + 1 = 0$

$\Rightarrow (\log_2 x - 1)^2 = 0 \Rightarrow \log_2 x = 1 \Rightarrow x = 2$

$S.S = \{2\}$

(11) $\log x - \log_x 100 = 1$

$\log x - \log_x 10^2 = 1 \Rightarrow \log x - 2\frac{1}{\log x} - 1 = 0$

$\Rightarrow (\log x)^2 - \log x - 2 = 0$

$\Rightarrow (\log x + 1)(\log x - 2) = 0$

$\log x = -1 \Rightarrow x = \frac{1}{10}$, $\log x = 2 \Rightarrow x = 100$

$S.S = \{0.1, 100\}$

(12) $(\log x)^3 = \log x^9$

$\Rightarrow (\log x)^3 - 9\log x = 0 \Rightarrow \log x\,[(\log x)^2 - 9] = 0$

$\Rightarrow \log x = 0 \Rightarrow x = 1$

or

$\log x = \pm 3 \Rightarrow x = 1000$ or $x = \frac{1}{1000}$

$S.S = \{1, 1000, 0.001\}$

(13) $x^{\log x} = 100x$ by taking Logarithms of the two sides

$\log x^{\log x} = \log 100x$

$\Rightarrow \log x \times \log x = \log 100x \Rightarrow (\log x)^2 = \log 100 + \log x$

$\Rightarrow (\log x)^2 - \log x - 2 = 0 \Rightarrow (\log x + 1)(\log x - 2) = 0$

$\Rightarrow x = 0.1$ or $x = 100 \Rightarrow S.S = \{0.1, 100\}$

(14) $\log_2 x + \log_4 x = -\frac{3}{2}$

$\frac{1}{\log_x 2} + \frac{1}{\log_x 4} = -\frac{3}{2} \Rightarrow \frac{1}{\log_x 2} + \frac{1}{2\log_x 2} = -\frac{3}{2}$

$\Rightarrow \frac{3}{2\log_x 2} = -\frac{3}{2} \Rightarrow \log_x 2 = -1 \Rightarrow x^{-1} = 2 \Rightarrow x = \frac{1}{2}$

$\therefore$ S.S $= \{\frac{1}{2}\}$

(15) $\sqrt{\log_2 x} = \log_2 \sqrt{x}$

Solution, $\sqrt{\log_2 x} = \frac{1}{2}\log_2 x$ by squaring

$\log_2 x = \frac{1}{4}(\log_2 x)^2$ $\times 4$

$\Rightarrow (\log_2 x)^4 - 4\log_2 x = 0$

$\log_2 x\,[\log_2 x - 4] = 0$

$\log_2 x = 0$	$\log_2 x = 4$
$x = 1$	$x = 2^4 = 16$

S.S $= \{1, 16\}$

If $x^2+y^2=8xy$, then prove that:

$$2\log(x+y)=1+\log x+\log y$$

Solution

$\because x^2+y^2=8xy$ by adding $2xy$

$\Rightarrow x^2+2xy+y^2=10xy \Rightarrow (x+y)^2=10xy$

by taking the logarithms of the two sides

$\Rightarrow \log(x+y)^2=\log 10xy$

$\Rightarrow 2\log(x+y)=\log 10+\log x+\log y$

$=1+\log x+\log y$

If $\log(x+y)=\frac{1}{2}(\log x+\log y)+\log 2$

then prove that $x=y$

Solution

$$\log(x+y)-\log 2=\frac{1}{2}\log xy$$

$$\Rightarrow \log\frac{x+y}{2}=\log(xy)^{\frac{1}{2}}$$

$$\frac{x+y}{2}=\sqrt{xy} \Rightarrow \frac{(x+y)^2}{4}=xy$$

$$(x^2+2xy+y^2=4xy \Rightarrow x^2-2xy+y^2=0$$

$$(x-y)^2=0 \Rightarrow x=y \;\#$$

Calculus

and

Trigonometry

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٣٦٢٣٦٦٨٢

Lesson 1:

Introduction to limits

Specified quantity:

It is the quantity which has determined result.

for example: 2×6 , $-2+7$, $\frac{8}{10}$, ...

Unspecified quantity:

It is the quantity which has no determined answer.

for example: $\frac{0}{0}$ is an unspecified

it has an infinite number of answers in $\mathbb{R}$

the product of any number $\times$ zero = zero

Undefined quantity:

It is the quantity which is meaningless

for example: $\frac{a}{0}$ where $a \in \mathbb{R} - \{0\}$

because there is no real number if multiplied by zero, the result will be a, $a \in \mathbb{R} - \{0\}$

Important notes:

① $\infty \pm a = \infty$, $-\infty \pm a = -\infty$

② $\infty \times a = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \\ \text{unspecified} & \text{at } a = 0 \end{cases}$

③ $-\infty \times a = \begin{cases} -\infty & \text{if } a > 0 \\ \infty & \text{if } a < 0 \\ \text{unspecified} & \text{if } a = 0 \end{cases}$

④ The unspecified quantities are seven

$\frac{\text{zero}}{\text{zero}}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $\infty \times \text{zero}$

, $(\text{zero})^{\text{zero}}$, $(\infty)^{\text{zero}}$, $(1)^{\infty}$

The concept of the limit of a function at a point

$$f(a^+) = \lim_{x \to a^+} f(x)$$

$$f(a^-) = \lim_{x \to a^-} f(x)$$

Definition:

If $f(a^+) = f(a^-) = l$,

then

$$\lim_{x \to a} f(x) = l$$

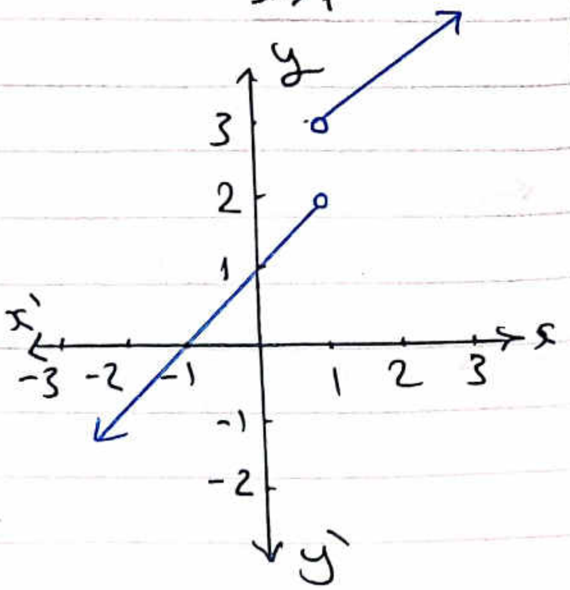
Remarks:

① At finding $\lim_{x \to a} f(x)$, it is not necessary that the function be defined at $x = a$, it should be defined only in an interval on the left of a and another interval on the right of a

② If $f(a^+) \neq f(a^-)$, then $\lim_{x \to a} f(x)$ doesnot exist

Example: from the opposite figure Find: (113)

(1) $f(1)$ (2) $f(1^+)$ (3) $f(1^-)$ (4) $\lim\limits_{x \to 1} f(x)$



Solution:

(1) $f(1)$ is undefined

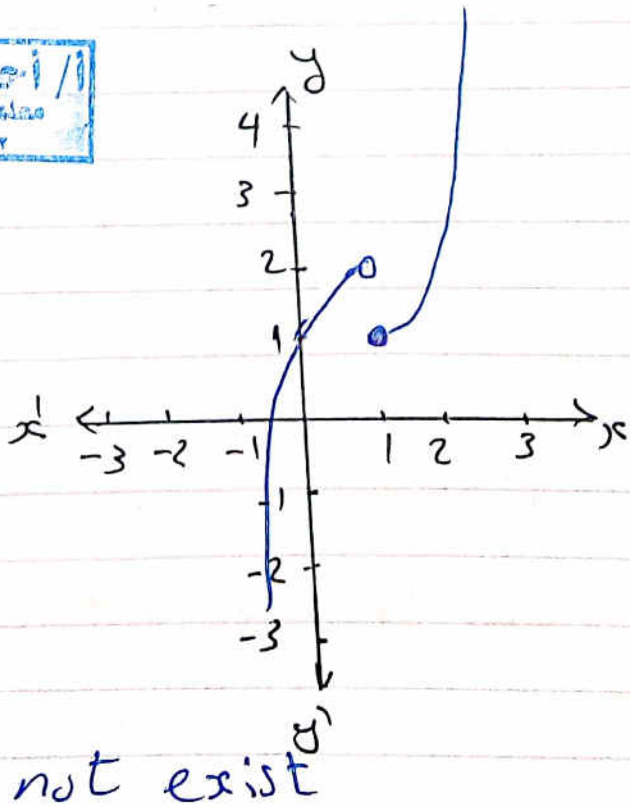
(2) $f(1^+) = \lim\limits_{x \to 1^+} f(x) = 3$

(3) $f(1^-) = \lim\limits_{x \to 1^-} f(x) = 2$

(4) $\because f(1^-) \neq f(1^+)$

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist

note: The Point has a jump dues to non existence a limit



Solution:

$f(1) = 1$

$f(1^-) = \lim\limits_{x \to 1^-} f(x) = 2$

$f(1^+) = \lim\limits_{x \to 1^+} f(x) = 1$

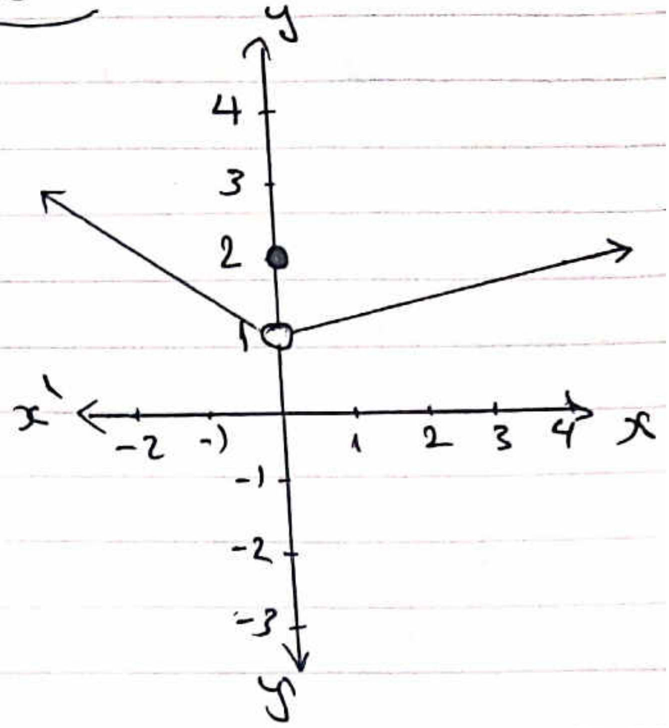
$\because f(1^-) \neq f(1^+)$

$\therefore \lim\limits_{x \to 1} f(x)$ does not exist

study the opposite figure; then find

(1) $f(0)$ ② $f(0^-)$

③ $f(0^+)$, ④ $\lim_{x \to 0} f(x)$



Solution:

(1) $f(0) = 2$

② $f(0^-) = \lim_{x \to 0^-} f(x) = 1$

③ $f(0^+) = \lim_{x \to 0^+} f(x) = 1$

④ $\because f(0^-) = f(0^+) = 1$

$\therefore \lim_{x \to 0} f(x) = 1$

notice that:

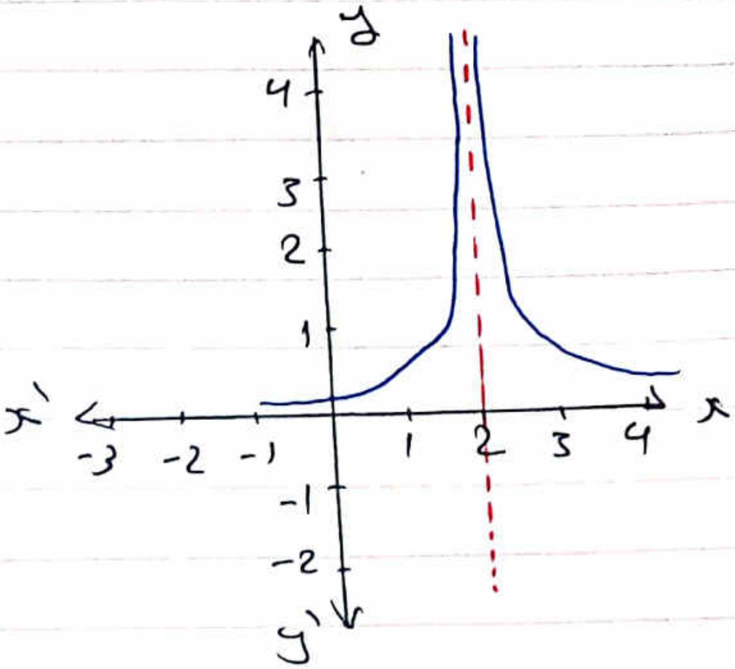
It is not necessary that the value of the function equals the value of the limit where

$f(0) \neq 2$, $\lim_{x \to 0} f(x) = 1$

from the opposite figure find if possible each of the following

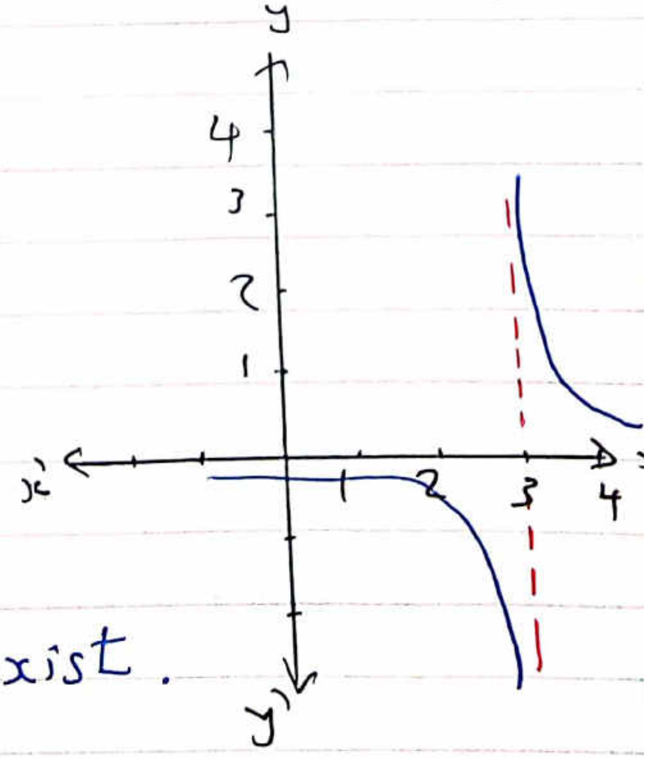
$f(3)$, $f(3^+)$, $f(3^-)$, $\lim_{x\to 3} f(x)$

Solution:



- $f(3)$ is undefined
- $f(3^+) = \lim_{x\to 3^+} f(x) = \infty$
- $f(3^-) = \lim_{x\to 3^-} f(x) = \infty$
- $\lim_{x\to 3} f(x) = \infty$

Solution



$f(3) =$ undefined

$f(3^+) = \lim_{x\to 3^+} f(x) = \infty$

$f(3^-) = \lim_{x\to 3^-} f(x) = -\infty$

$\therefore \lim_{x\to 3} f(x)$ does not exist.

Scanned by CamScanner

Lesson 2 : Finding the limit of a function algebraically

to find the of a rational function $f(x)$

we use the direct substitution

If : $f(a) = \dfrac{\text{zero}}{\text{zero}}$

this means that $(x-a)$ is a common factor between the numerator and the denominator

In this case to find $\lim\limits_{x \to a} f(x)$

we cancel the factor $(x-a)$ using factorization or long division

Example: Find $\lim\limits_{x \to 4} \dfrac{x^2-16}{x-4}$

Solution: using the direct substitution

$$\lim_{x \to 4} \frac{x^2-16}{x-4} = \frac{16-16}{4-4} = \frac{\text{zero}}{\text{zero}}$$

$\therefore$ by factorizing $\lim\limits_{x \to 4} \dfrac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}} = \lim\limits_{x \to 4} x+4$

$= 4+4 = 8$

Lesson 2 : Finding the limit of a function algebraically

to find the of a rational function $f(x)$

we use the direct substitution

If : $f(a) = \frac{\text{zero}}{\text{zero}}$

this means that $(x-a)$ is a common factor between the numerator and the denominator

In this case to find $\lim\limits_{x \to a} f(x)$

we cancel the factor $(x-a)$ using

factorization or long division

Example: Find $\lim\limits_{x \to 4} \frac{x^2-16}{x-4}$

Solution: using the direct substitution

$$\lim_{x \to 4} \frac{x^2-16}{x-4} = \frac{16-16}{4-4} = \frac{\text{zero}}{\text{zero}}$$

$\therefore$ by factorizing $\lim\limits_{x \to 4} \frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}} = \lim\limits_{x \to 4} x+4$

$= 4+4 = 8$

Example: Find Each of the following: $\boxed{117}$

$\boxed{1}$ $\displaystyle \lim_{x \to 4} \frac{2x-8}{x^2-x-12} = \lim_{x \to 4} \frac{2\cancel{(x-4)}}{(x+3)\cancel{(x-4)}}$

$\displaystyle = \lim_{x \to 4} \frac{2}{x+3} = \frac{2}{7}$

$\boxed{2}$ $\displaystyle \lim_{x \to -3} \frac{x^2+4x+3}{x^2-9} = \lim_{x \to -3} \frac{(x+1)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}}$

$\displaystyle = \lim_{x \to -3} \frac{x+1}{x-3} = \frac{-2}{-6} = \frac{1}{3}$

$\boxed{3}$ $\displaystyle \lim_{x \to 9} \frac{9-x}{x^2-81} = \lim_{x \to 9} \frac{-\cancel{(x-9)}}{(x+9)\cancel{(x-9)}}$

$\displaystyle = \lim_{x \to 9} \frac{-1}{x+9} = \frac{-1}{18}$

$\boxed{4}$ $\displaystyle \lim_{x \to 0} \frac{(2x-1)^2-1}{5x} = \lim_{x \to 0} \frac{(2x-1+1)(2x-1-1)}{5x}$

$\displaystyle = \lim_{x \to 0} \frac{(2x-2)(2\cancel{x})}{5\cancel{x}} = \frac{-4}{5}$

$\boxed{5}$ $\displaystyle \lim_{x \to -2} \frac{x+2}{x^4-16}$

$$\begin{array}{r|rrrrr} -2 & 1 & 0 & 0 & 0 & -16 \\ & & -2 & 4 & -8 & 16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

① ② ③ ④

$\displaystyle = \lim_{x \to -2} \frac{\cancel{(x+2)}}{\cancel{(x+2)}(x^3-2x^2+4x-8)}$

$\displaystyle = \lim_{x \to -2} \frac{1}{(-2)^3-2(-2)^2+4\times -2-8} = -\frac{1}{32}$

6) $$\lim_{x \to 9} \frac{x+\sqrt{x}-12}{x-9} = \lim_{x \to 9} \frac{\cancel{(\sqrt{x}-3)}(\sqrt{x}+4)}{\cancel{(\sqrt{x}-3)}(\sqrt{x}+3)}$$

$$= \lim_{x \to 9} \frac{\sqrt{x}+4}{\sqrt{x}+3} = \frac{7}{6}$$

7) $$\lim_{x \to 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$$

$$= \lim_{x \to 0} \frac{\frac{2-2-x}{2(x+2)}}{x} = \lim_{x \to 0} \frac{-\cancel{x}}{2\cancel{x}(x+2)}$$

$$= \lim_{x \to 0} \frac{-1}{2(x+2)} = -\frac{1}{4}$$

8) $$\lim_{x \to 1} \left(\frac{1}{x-1} - \frac{3}{x^3-1} \right)$$

$$= \lim_{x \to 1} \left(\frac{1}{x-1} - \frac{3}{(x-1)(x^2+x+1)} \right)$$

$$= \lim_{x \to 1} \frac{x^2+x+1-3}{(x-1)(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{x^2+x-2}{(x-1)(x^2+x+1} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)}(x^2+x+1)}$$

$$= \lim_{x \to 1} \frac{x+2}{x^2+x+1}$$

$$= \frac{3}{3} = 1$$

9) $\lim\limits_{x \to -2} \dfrac{x^2+4x+4}{x^3+x^2-8x-12}$

$f(-2) = \dfrac{\text{Zero}}{\text{Zero}}$

$\therefore (x+2)$ is a common factor between numerator and denominator

we devide the denominator by $(x+2)$ by using:

① long division $\Rightarrow$

$$\begin{array}{r|l} x+2 & x^3+x^2-8x-12 \\ \hline x^2-x-6 & \underline{-x^3 - 2x^2} \\ & -x^2-8x \\ & \underline{+x^2 + 2x} \\ & -6x-12 \\ & \underline{+6x + 12} \\ & 0 \quad 0 \end{array}$$

② synthetic division $\Downarrow$

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -8 & -12 \\ & \downarrow & -2 & 2 & 12 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$\therefore \lim\limits_{x \to -2} \dfrac{\cancel{(x+2)}(x+2)}{\cancel{(x+2)}(x^2-x-6)} = \lim\limits_{x \to -2} \dfrac{x+2}{x^2-x-6} = \dfrac{\text{Zero}}{\text{Zero}}$

$= \lim\limits_{x \to -2} \dfrac{\cancel{(x+2)}}{\cancel{(x+2)}(x-3)} = \dfrac{-1}{5}$

In case of existence of a difference of two square roots of algebraic expression in numerator or denominator or both we multiply each of the numerator and denominator by the conjugate of the numerator or the denominator or both when the result of the direct substitution equals $\frac{zero}{zero}$

Example: Find $\lim\limits_{x \to 5} \frac{\sqrt{x-1}-2}{x-5}$

$$\lim_{x \to 5} \frac{\sqrt{x-1}-2}{x-5} = \frac{zero}{zero}$$

multiplying by the conjugate of the numerator.

$$\lim_{x \to 5} \frac{(\sqrt{x-1}-2)(\sqrt{x-1}+2)}{(x-5)(\sqrt{x-1}+2)} = \lim_{x \to 5} \frac{x-1-4}{(x-5)(\sqrt{x-1}+2)}$$

$$= \lim_{x \to 5} \frac{\cancel{(x-5)}}{\cancel{(x-5)}(\sqrt{x-1}+2)} = \lim_{x \to 5} \frac{1}{\sqrt{x-1}+2}$$

$$= \frac{1}{4}$$

Example: Find each of the following: $\boxed{121}$

$\boxed{1}$ $\lim\limits_{x \to -1} \dfrac{x+1}{\sqrt{x+5}-2}$

- multiplying by the Conjugate of denominator

$$= \lim_{x \to -1} \frac{(x+1)(\sqrt{x+5}+2)}{(\sqrt{x+5}-2)(\sqrt{x+5}+2)}$$

$$= \lim_{x \to -1} \frac{\cancel{(x+1)}(\sqrt{x+5}+2)}{(\cancel{x+5-4})}$$

$$= \lim_{x \to -1} \sqrt{x+5}+2 = 4$$

$\boxed{2}$ $\lim\limits_{x \to 3} \dfrac{\sqrt{x+1}-2}{\sqrt{x-2}-1}$

Multiplying by the Conjugate of the denominator and numerator

$$\lim_{x \to 3} \frac{(\sqrt{x+1}-2)(\sqrt{x+1}+2)(\sqrt{x-2}+1)}{(\sqrt{x-2}-1)(\sqrt{x-2}+1)(\sqrt{x+1}+2)}$$

$$= \lim_{x \to 3} \frac{(\cancel{x+1-4})(\sqrt{x-2}+1)}{(\cancel{x-2-1})(\sqrt{x+1}+2)}$$

$$= \lim_{x \to 3} \frac{\sqrt{x-2}+1}{\sqrt{x+1}+2} = \frac{2}{4} = \frac{1}{2}$$

If $\lim\limits_{x \to -1} \dfrac{x^2-(a-1)x-a}{x+1} = 4$, then find a

Solution; $\because$ $(x+1)$ is a common factor

$\therefore \lim\limits_{x \to -1} \dfrac{(x+1)(x-a)}{(x+1)} = \lim\limits_{x \to -1} (x-a) = -1-a$

$\therefore -1-a = 4 \Rightarrow a = -5$

If $\lim\limits_{x \to 1} \left(\dfrac{x^2+ax+b}{x-1} \right) = 5$

find the value of a and b

Solution: $\because$ $(x-1)$ is a common factor

$\lim\limits_{x \to 1} \dfrac{\cancel{(x-1)}(x-k)}{\cancel{(x-1)}} = \lim\limits_{x \to 1} (x-k) = k+1$

$\therefore -k+1 = 5 \Rightarrow k = -4$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٢٦٣٦٦٨٢

$\Rightarrow x^2 + ax + b = (x-1)(x+4)$

$= x^2 - x + 4x - 4$

$\Rightarrow x^2 + ax + b = x^2 + 3x - 4$

Comparing the Coefficient

$\therefore$ $\boxed{a = 3}$, $\boxed{b = -4}$

Lesson 3: Theorem (4) "The law"

Theorem 4:

for every $n \in \mathbb{R} - \{0\}$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

Corollaries

1) $$\lim_{x \to 0} \frac{(x+a)^n - a^n}{x} = n a^{n-1}$$

2) $$\lim_{x \to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} (a)^{n-m}$$

where $n \in \mathbb{R} - \{0\}$, $m \in \mathbb{R} - \{0\}$

Find each of the following:

(1) $\lim\limits_{x \to 2} \dfrac{x^5 - 32}{x^3 - 8} = \lim\limits_{x \to 2} \dfrac{x^5 - 2^5}{x^3 - 2^3} = \dfrac{5}{3}(2)^{5-3} = \dfrac{20}{3}$

(2) $\lim\limits_{x \to -5} \dfrac{x^4 - 625}{x + 5} = \lim\limits_{x \to -5} \dfrac{x^4 - (-5)^4}{x - (-5)} = 4(-5)^{4-1} = -500$

(3) $\lim\limits_{x \to -3} \dfrac{x^4 - 81}{x^5 + 243} = \lim\limits_{x \to -3} \dfrac{x^4 - (-3)^4}{x^5 - (-3)^5} = \dfrac{4}{5}(-3)^{4-5} = -\dfrac{4}{15}$

(4) $\lim\limits_{x \to -\frac{1}{2}} \dfrac{32x^5 + 1}{64x^6 - 1} = \lim\limits_{x \to -\frac{1}{2}} \dfrac{32\left(x^5 + \frac{1}{32}\right)}{64\left(x^6 - \frac{1}{64}\right)}$

$= \dfrac{32}{64} \lim\limits_{x \to -\frac{1}{2}} \dfrac{x^5 - \left(-\frac{1}{2}\right)^5}{x^6 - \left(-\frac{1}{2}\right)^6}$

$= \dfrac{1}{2} \times \dfrac{5}{6} \times \left(-\dfrac{1}{2}\right)^{5-6} = -\dfrac{5}{6}$

Another Solution

$\lim\limits_{x \to -\frac{1}{2}} \dfrac{32x^5 + 1}{64x^6 - 1} = \lim\limits_{2x \to -1} \dfrac{(2x)^5 - (-1)^5}{(2x)^6 - (-1)^6} = \dfrac{5}{6}(-1)^{5-6}$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٣١٣٦٦٨٢

$= -\dfrac{5}{6}$

(5) $\lim\limits_{x\to 3/2} \dfrac{16x^5-81x}{2x^3-3x^2} = \lim\limits_{2x\to 3} \dfrac{x(16x^4-81)}{x^2(2x-3)}$

$= \lim\limits_{2x\to 3} \dfrac{(2x)^4-(3)^4}{x[2x-3]} = \dfrac{1}{3/2} \times 4 \times (3)^{4-1}$

$= 72$

(6) $\lim\limits_{x\to 2} \dfrac{x^{-8}-(16)^{-2}}{x-2} = \lim\limits_{x\to 2} \dfrac{x^{-8}-(2)^{-8}}{x-2} = -8(2)^{-8-1}$

$= -\dfrac{1}{64}$

(7) $\lim\limits_{x\to 1} \dfrac{\sqrt[7]{x}-1}{x-1} = \lim\limits_{x\to 1} \dfrac{x^{1/7}-1^{1/7}}{x-1} = \dfrac{1}{7}(1)^{\frac{1}{7}-1} = \dfrac{1}{7}$

(8) $\lim\limits_{x\to 16} \dfrac{\sqrt[4]{x^7}-128}{x-16} = \lim\limits_{x\to 16} \dfrac{x^{7/4}-(16)^{7/4}}{x-16} = \dfrac{7}{4}(16)^{\frac{7}{4}-1}$

$= 14$

(9) $\lim\limits_{x\to 1} \dfrac{x^{21/2}-x^{1/2}}{x^{14/3}-x^{2/3}} = \lim\limits_{x\to 1} \dfrac{x^{1/2}[x^{10}-1]}{x^{2/3}[x^4-1]}$

$= \lim\limits_{x\to 1} x^{-1/6} \dfrac{x^{10}-1^{10}}{x^4-1^4}$

$= (1)^{-1/6} \times \dfrac{10}{4}(1)^{10-4} = \dfrac{5}{2}$

(10) $\lim\limits_{x \to 2} \dfrac{x^{10} - 1024}{x^2 - 3x + 2} = \lim\limits_{x \to 2} \dfrac{x^{10} - 2^{10}}{(x-1)(x-2)}$

$= \lim\limits_{x \to 2} \dfrac{1}{x-1} \times \dfrac{x^{10} - 2^{10}}{x-2} = \dfrac{1}{2-1} \times \dfrac{10}{1}(2)^{10-1}$

$= 5120$

(11) $\lim\limits_{x \to 6} \dfrac{(x-5)^7 - 1}{x-6} = \lim\limits_{x-5 \to 1} \dfrac{(x-5)^7 - 1^7}{(x-5) - 1} = 7(1)^{7-1} = 7$

(12) $\lim\limits_{x \to -2} \dfrac{(x+3)^5 - 1}{x+2} = \lim\limits_{x+3 \to 1} \dfrac{(x+3)^5 - 1}{(x+3) - 1} = 5(-1)^{5-1} = 5$

(13) $\lim\limits_{h \to 0} \dfrac{(3+h)^4 - 81}{6h} = \dfrac{1}{6} \lim\limits_{h \to 0} \dfrac{(h+3)^4 - 3^4}{h}$

$= \dfrac{1}{6} \times 4(3)^{4-1} = 18$

(14) $\lim\limits_{x \to 0} \dfrac{(1-2x)^5 - 1}{5x} = \dfrac{-2}{5} \lim\limits_{-2x \to 0} \dfrac{(-2x+1)^5 - 1^5}{-2x}$

$= \dfrac{-2}{5} \times 5(1)^{5-1} = -2$

(15) $\lim\limits_{h \to 0} \dfrac{(x-2h)^{17} - x^{17}}{51h} = \dfrac{-2}{51} \lim\limits_{h \to 0} \dfrac{(-2h+x)^{17} - x^{17}}{-2h}$

$= \dfrac{-2}{51} \times 17(x)^{17-1}$

$= \dfrac{-2}{3} x^{16}$ #

Scanned by CamScanner

Another solution for 11 , 12

11) $$\lim_{x \to 6} \frac{(x-5)^7 - 1}{x-6} = \lim_{x-6 \to 0} \frac{[(x-6)+1]^7 - 1^7}{(x-6)}$$

$$= 7(1)^{7-1} = 7$$

12) $$\lim_{x \to -2} \frac{(x+3)^5 - 1}{x+2} = \lim_{x+2 \to 0} \frac{[(x+2)+1]^5 - 1^5}{(x+2)} = 5(1)^4 = 5$$

16) $$\lim_{x \to 7} \frac{\sqrt[5]{x+25} - 2}{x-7} = \lim_{x-7 \to 0} \frac{[(x-7)+32]^{\frac{1}{5}} - (32)^{\frac{1}{5}}}{(x-7)}$$

$$= \frac{1}{5}(32)^{\frac{1}{5}-1} = \frac{1}{5}(32)^{-\frac{4}{5}} = \frac{1}{80}$$

17) $$\lim_{x \to 2} \frac{(x-4)^5 + 32}{x-2} = \lim_{x-2 \to 0} \frac{[(x-2)+(-2)]^5 - (-2)^5}{(x-2)}$$

$$= 5(-2)^4 = 80$$

18) $$\lim_{x \to -2} \frac{(x+3)^5 - 1}{x^2 - 4} = \lim_{x+2 \to 0} \frac{[(x+2)+1]^5 - 1^5}{(x-2)(x+2)}$$

$$= \lim_{x+2 \to 0} \frac{1}{x-2} \times \frac{[(x+2)+1]^5 - 1^5}{(x+2)}$$

$$= \frac{1}{-2-2} \times 5(1)^4 = -\frac{5}{4}$$

19) $$\lim_{x\to 2} \frac{(x-4)^5+32}{x-2}$$

$$= \lim_{x-2\to 0} \frac{[(x-2)+(-2)]^5-(-2)^5}{(x-2)} = 5(-2)^4 = 80$$

20) $$\lim_{x\to 1} \frac{x^{19}+x^8-2}{x-1} = \lim_{x\to 1} \frac{x^{19}-1+x^8-1}{x-1}$$

$$= \lim_{x\to 1} \frac{x^{19}-1}{x-1} + \lim_{x\to 1} \frac{x^8-1}{x-1} = 19+8 = 27$$

21) $$\lim_{x\to 1} \frac{\sqrt{x}+\sqrt[3]{x}-2}{x-1} = \lim_{x\to 1} \frac{x^{\frac{1}{2}}-1+x^{\frac{1}{3}}-1}{x-1}$$

$$= \lim_{x\to 1} \frac{x^{\frac{1}{2}}-1^{\frac{1}{2}}}{x-1} + \lim_{x\to 1} \frac{x^{\frac{1}{3}}-1^{\frac{1}{3}}}{x-1}$$

$$= \tfrac{1}{2}(1)^{\frac{1}{2}-1} + \tfrac{1}{3}(1)^{\frac{1}{3}-1} = \tfrac{1}{2}+\tfrac{1}{3} = \tfrac{5}{6}$$

22) $$\lim_{x\to 2} \frac{x^5+x^2-36}{x-2} = \lim_{x\to 2} \frac{x^5-32+x^2-4}{x-2}$$

$$= \lim_{x\to 2} \frac{x^5-2^5}{x-2} + \lim_{x\to 2} \frac{x^2-2^2}{x-2}$$

$$= 5(2)^4 + 2(2)^1 = 84$$

Scanned by CamScanner

Lesson 4:

Limit of the function at infinity

Theorem S

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Corollaries

If $a \in \mathbb{R}$, then:

$\boxed{1}$ $\lim_{x \to \infty} \frac{a}{x} = \text{zero}$ (2) $\lim_{x \to \infty} \frac{a}{x^n} = \text{zero}$, $n \in \mathbb{R}^+$.

* Getting the limit of a rational function at infinity:

If the direct substitution by $x = \infty$ gives $\frac{\infty}{\infty}$ we devide each of numerator and denominator by x raised to the higher power in the denominator

Remark

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} =$$

(1) a real number not equal to zero if the degree of the numerator = the degree of denominator

(2) zero if the degree of numerator < the degree of denominator

(3) the limit $= \pm\infty$ if the degree of numerator > the degree of denominator

---

Example

1. $\lim\limits_{x \to \infty} \frac{3}{x^2} = \text{zero}$

2. $\lim\limits_{x \to \infty} \left(\frac{3}{x^2} - 2\right) = (\text{zero} - 2) = -2$

3. $\lim\limits_{x \to \infty} \frac{2x-5}{3x+8}$

dividing both numerator and denominator by x

$$\lim_{x \to \infty} \frac{2 - \frac{5}{x}}{3 + \frac{8}{x}} = \frac{2-0}{3+0} = \frac{2}{3}$$

note that:

degree of the numerator = degree of the denominator

4. $\lim\limits_{x \to \infty} \frac{5 - 6x - 3x^2}{2x^2 + x + 4}$

dividing both numerator and denominator by x^2

$$= \lim_{x \to \infty} \frac{\frac{5}{x^2} - \frac{6}{x} - 3}{2 + \frac{1}{x} + \frac{4}{x^2}} = \frac{0 - 0 - 3}{2 + 0 + 0} = \frac{-3}{2}$$

(5) $\lim_{x \to \infty} \frac{x^3-2}{|x|^3+1}$

Solution: when $x \to \infty \Rightarrow |x| = x$

$$= \lim_{x \to \infty} \frac{x^3-2}{x^3+1} = \lim_{x \to \infty} \frac{1-\frac{2}{x^3}}{1+\frac{1}{x^3}} = \frac{1-0}{1+0} = 1$$

(6) $\lim_{x \to \infty} (x^3+5x^2+1)$

$$= \infty + \infty + 1 = \infty$$

(7) $\lim_{x \to \infty} (5+x-x^2) = 5+\infty-\infty$ unspecified quantity

$$= \lim_{x \to \infty} x^2\left(\frac{5}{x^2}-\frac{1}{x}-1\right)$$

$$= \infty(0-0-1) = \infty \times -1 = -\infty$$

(8) $\lim_{x \to \infty} \frac{x^3-4x+5}{(2x-1)^3}$

dividing by x^3

$$\Rightarrow \lim_{x \to \infty} \frac{1-\frac{4}{x^2}+\frac{5}{x^3}}{(2-\frac{1}{x})^3} = \frac{1-0+0}{(2-0)^3} = \frac{1}{8}$$

9) $$\lim_{x\to\infty} \frac{(x+2)^3(3-2x^2)}{3x(x^2+7)^2}$$

dividing both numerator and denominator by x^5

$$\Rightarrow \lim_{x\to\infty} \frac{(1+\frac{2}{x})^3(\frac{3}{x}-2)}{3(1+\frac{7}{x^2})^2} = \frac{(1+0)^3(-2)}{3(1)^2} = \frac{-2}{3}$$

10) $$\lim_{x\to\infty} \frac{1}{x}\sqrt{3+4x^2} = \lim_{x\to\infty} \frac{\sqrt{3+4x^2}}{x}$$

dividing by $x = \sqrt{x^2}$

$$\Rightarrow \lim_{x\to\infty} \frac{\sqrt{\frac{3}{x^2}+4}}{1} = 2$$

11) $$\lim_{x\to\infty} \frac{4-3x^3}{\sqrt{x^6+9}}$$ dividing by $\sqrt{x^6} = x^3$

$$\Rightarrow \lim_{x\to\infty} \frac{\frac{4}{x^3}-3}{\sqrt{1+\frac{9}{x^6}}} = \frac{0-3}{\sqrt{1+0}} = -3$$

12) $$\lim_{x\to\infty} \frac{\sqrt[4]{x^6+8} - \sqrt[6]{x^2+4}}{\sqrt{x^3+9}}$$

by dividing by $\sqrt{x^3} = \sqrt[6]{x^9} = \sqrt[4]{x^6}$

$$\Rightarrow \lim_{x\to\infty} \frac{\sqrt[4]{1+\frac{8}{x^6}} - \sqrt{\frac{1}{x^7}+\frac{4}{x^9}}}{\sqrt{1+\frac{9}{x^3}}} = \frac{1-0}{1} = 1$$

(13) $$\lim_{x\to\infty} \frac{5+x^{-2}}{3x^{-2}+1} = \lim_{x\to\infty} \frac{5+\frac{1}{x^2}}{\frac{3}{x^2}+1} = 5$$

(14) $$\lim_{x\to\infty} \left(7+\frac{2x^2}{(x+3)^2}\right)$$

$$= \lim_{x\to\infty} 7 + \lim_{x\to\infty} \frac{2x^2}{(x+3)^2}$$

$$= \lim_{x\to\infty} 7 + \lim_{x\to\infty} \frac{2}{(1+\frac{3}{x})^2} = 7+2=9$$

(15) $$\lim_{x\to\infty} \left(\frac{x}{2x+1}+\frac{3x^2}{(x-3)^2}\right)$$

$$= \lim_{x\to\infty} \frac{x}{2x+1} + \lim_{x\to\infty} \frac{3x^2}{(x-3)^2}$$

$$= \lim_{x\to\infty} \frac{1}{2+\frac{1}{x}} + \lim_{x\to\infty} \frac{3}{(1-\frac{3}{x})^2} = \frac{1}{2}+3 = \frac{7}{2}$$

(16) $$\lim_{x\to\infty} \left(\frac{2}{3}-\frac{3x}{2x+7}\right)$$

$$= \lim_{x\to\infty} \frac{2}{3} - \lim_{x\to\infty} \frac{3x}{2x+7}$$

$$= \lim_{x\to\infty} \frac{2}{3} - \lim_{x\to\infty} \frac{3}{2+\frac{7}{x}} = \frac{2}{3}-\frac{3}{2} = -\frac{5}{6}$$

(17) $\lim_{x \to \infty} \left(\sqrt{x^2-2} - \sqrt{x^2+x} \right)$

multiplying by the conjugate

$$\Rightarrow \lim_{x \to \infty} \frac{\left(\sqrt{x^2-2} - \sqrt{x^2+x}\right)\left(\sqrt{x^2-2} + \sqrt{x^2+x}\right)}{\left(\sqrt{x^2-2} + \sqrt{x^2+x}\right)}$$

$$= \lim_{x \to \infty} \frac{(x^2-2) - (x^2+x)}{\left(\sqrt{x^2-2} + \sqrt{x^2+x}\right)} =$$

$$= \lim_{x \to \infty} \frac{-2-x}{\left(\sqrt{x^2-2} + \sqrt{x^2+x}\right)}$$ dividing by $x = \sqrt{x^2}$

$$\Rightarrow \lim_{x \to \infty} \frac{\frac{-2}{x} - 1}{\left(\sqrt{1-\frac{2}{x^2}} + \sqrt{1+\frac{1}{x}}\right)} = \frac{-1}{1+1} = -1/2.$$

(18) $\lim_{x \to \infty} \left(\frac{3x^2-5x+1}{4x^2-7} \right)^{\frac{1}{x}}$

$$= \lim_{x \to \infty} \left(\frac{3 - \frac{5}{x} + \frac{1}{x^2}}{4 - \frac{7}{x^2}} \right)^{\frac{1}{x}}$$

$$= \left(\frac{3}{4}\right)^0 = 1$$

(19) Find the value of each of a and n

if $\lim\limits_{x \to \infty} \dfrac{4ax^n - 4x + 5}{3 - 9x + 8x^2} = 3$

Solution:

$\because \lim\limits_{x \to \infty} \dfrac{4ax^n - 4x + 5}{3 - 9x + 8x^2} = 3$

$\therefore$ the degree of numerator = degree of the denominator

$\therefore n = 2$

$\Rightarrow \lim\limits_{x \to \infty} \dfrac{4ax^2 - 4x + 5}{3 - 9x + 8x^2} = 3$ (Dividing by x^2)

$\Rightarrow \lim\limits_{x \to \infty} \dfrac{4a - \frac{4}{x} + \frac{5}{x^2}}{\frac{3}{x^2} - \frac{9}{x} + 8} = 3 \Rightarrow \dfrac{4a}{8} = 3$

$\Rightarrow 4a = 24 \Rightarrow \boxed{a = 6}$

(20) Find the value of each of a and b if

$\lim\limits_{x \to \infty} \dfrac{x^3 + 2x + 1}{(a+1)x^4 + (7-b)x^3 + x^2} = \infty$

$\because$ the limit $= \infty$ $\therefore$ the degree of the numerator > the degree of the denominator

$\therefore a + 1 = 0 \Rightarrow a = -1$

$, \; 7 - b = 0 \Rightarrow a = 7$

(21)

$$\lim_{x\to\infty} \frac{x^{-2}+x^{-4}}{(2x+3)^{-2}}$$

dividing both the denominator and numerator by x^{-2}

$$\Rightarrow \lim_{x\to\infty} \frac{1+\frac{1}{x^2}}{\left(2+\frac{3}{x}\right)^{-2}} = \frac{1}{2^{-2}} = 4$$

(22) $$\lim_{n\to\infty} n\left[\left(a+\frac{1}{n}\right)^4 - a^4\right]$$

$$= \lim_{n\to\infty} \frac{\left[\left(a+\frac{1}{n}\right)^4 - a^4\right]}{\frac{1}{n}} =$$

$$= \lim_{\frac{1}{n}\to 0} \frac{\left(a+\frac{1}{n}\right)^4 - a^4}{\frac{1}{n}} = 4\,a^{4-1} = 4a^3$$

(23) $$\lim_{x\to\infty} x\left[\left(3+\frac{1}{x}\right)^5 - 243\right]$$

$$= \lim_{\frac{1}{x}\to 0} \frac{\left(\frac{1}{x}+3\right)^5 - 3^5}{\frac{1}{x}} = 5(3)^4 = 405$$

(24) $\lim\limits_{x \to \infty} (2x^{-1} - x^{-2}) \sqrt{4x^2+1}$

$$= \lim_{x \to \infty} \left(\frac{2}{x} - \frac{1}{x^2} \right) \sqrt{4x^2+1}$$

$$= \lim_{x \to \infty} \frac{2\sqrt{4x^2+1}}{x} - \lim_{x \to \infty} \frac{\sqrt{4x^2+1}}{x^2}$$

$$= \lim_{x \to \infty} \frac{2\sqrt{4+\frac{1}{x^2}}}{1} - \lim_{x \to \infty} \frac{\sqrt{\frac{4}{x^2}+\frac{1}{x^4}}}{1} = 4 - 0 = 4$$

(25) $\lim\limits_{x \to \infty} \dfrac{x\sqrt{x} + 2\sqrt{x-1}}{x+5-\sqrt{x^3-x}}$

Dividing both numerator and denominator by $\sqrt{x^3} = x\sqrt{x}$

$$\Rightarrow \lim_{x \to \infty} \frac{1 + 2\sqrt{\frac{1}{x^2} - \frac{1}{x^3}}}{\frac{1}{\sqrt{x}} + \frac{5}{x\sqrt{x}} - \sqrt{1 - \frac{1}{x^2}}} = \frac{1+0}{0+0-1} = -1$$

أ/ أحمد عمر
معلم أول رياضيات

Lesson 5:
Limits of trigonometric functions

Theorem

If x is the measure of an angle in radians, then:

(1) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$ (2) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

Corollary (1)

(1) $\lim\limits_{x \to 0} \dfrac{\sin ax}{x} = a$, then $\lim\limits_{x \to 0} \dfrac{\sin ax}{bx} = \dfrac{a}{b}$

(2) $\lim\limits_{x \to 0} \dfrac{\tan ax}{x} = a$, then $\lim\limits_{x \to 0} \dfrac{\tan ax}{bx} = \dfrac{a}{b}$

where x is in radian measure

Remark

$\lim\limits_{x \to 0} \dfrac{\sin ax^2}{x^2} = a$ but

$$\lim_{x \to 0} \frac{\sin^2 ax}{x^2} = \lim_{x \to 0} \left(\frac{\sin ax}{x}\right)^2$$

$$= a^2$$

Corollary (2)

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Remark

If x is the measure of an angle in degree measure, then:

(1) $\lim_{x \to 0} \frac{\sin x}{x} = \frac{\pi}{180}$ (2) $\lim_{x \to 0} \frac{\tan x}{x} = \frac{\pi}{180}$

Example: Find each of the following:

1) $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

2) $\lim_{x \to 0} \frac{\tan 3x}{\frac{x}{3}} = \frac{3}{\frac{1}{3}} = 9.$

3) $\lim_{x \to 0} \frac{\sin 3x - \sin 2x}{5x} = \lim_{x \to 0} \frac{\sin 3x}{5x} - \lim_{x \to 0} \frac{\sin 2x}{5x}$

$$= \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$$

Another solution

$$= \lim \frac{\frac{\sin 3x}{x} - \frac{\sin 2x}{x}}{\frac{5x}{x}} = \frac{3-2}{5} = \frac{1}{5}$$

④ $\lim\limits_{x \to 0} \dfrac{x + x\cos x}{\sin x \cos x}$

dividing both numerator and denominator by x

$$= \lim_{x \to 0} \frac{1 + \cos x}{\frac{\sin x}{x} \cos x} = \frac{1+1}{1 \times 1} = 2$$

$\cos 0 = 1$

⑤ $\lim\limits_{x \to 0} \dfrac{\sin 24x \times \cos 6x}{\tan 6x \times \cos 24x}$

dividing both numerator and denominator by x

$$= \lim_{x \to 0} \frac{\frac{\sin 24x}{x} \times \cos 6x}{\frac{\tan 6x}{x} \times \cos 24x} = \frac{24 \times 1}{6 \times 1} = 4$$

⑥ $\lim\limits_{x \to \frac{1}{2}} \dfrac{x \cos(-2x+1)}{x^2 + x}$

$$= \lim_{x \to \frac{1}{2}} \frac{\cancel{x} \cos(-2x+1)}{\cancel{x}(x+1)} = \lim_{x \to \frac{1}{2}} \frac{\cos(-2x+1)}{x+1}$$

$$= \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

⑦ $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x}$

$$= \lim_{x \to 0} x \frac{\sin^2 x}{x^2} = \lim_{x \to 0} x \left(\frac{\sin x}{x}\right)^2 = 0 \times 1 = 0$$

8) $\lim\limits_{x\to 0} \dfrac{x\tan 2x}{x^2+\sin^2 3x}$

dividing both numerator and denominator by x^2

$$= \lim_{x\to 0} \frac{\frac{\tan 2x}{x}}{1+\frac{\sin^2 3x}{x^2}} = \lim_{x\to 0} \frac{\frac{\tan 2x}{x}}{1+\left(\frac{\sin 3x}{x}\right)^2} = \frac{2}{1+3^2} = \frac{1}{5}$$

9) $\lim\limits_{x\to 0} \dfrac{\tan 3x^2+\sin^2 5x}{x^2}$

$$= \lim_{x\to 0} \frac{\tan 3x^2}{x^2} + \lim_{x\to 0} \frac{\sin^2 5x}{x^2}$$

$$= 3 + 5^2 = 28$$

10) $\lim\limits_{x\to 0} \dfrac{\sin 5x^3+\sin^3 5x}{2x^3}$

$$= \lim_{x\to 0} \frac{\sin 5x^3}{2x^3} + \lim_{x\to 0} \frac{1}{2}\times\left(\frac{\sin 5x}{x}\right)^3$$

$$= \frac{5}{2} + \frac{1}{2}\times 5^3 = \frac{130}{2} = 65$$

11) $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x^2+x-2} = \lim\limits_{x-1\to 0} \dfrac{(\sin(x-1)}{(x+2)(x-1)}$

$$= \lim_{x-1\to 0} \frac{1}{x+2} \times \frac{\sin(x-1)}{(x-1)} = \frac{1}{3}\times 1 = \frac{1}{3}$$

(12) $\lim_{x \to 0} \frac{3(1-\cos x)}{x} = 3 \times 0 = 0$

(13) $\lim_{x \to 0} \frac{\cos x - 1}{\sin x} =$

$= \lim_{x \to 0} \frac{\cos x - 1}{x} \times \frac{x}{\sin x} = 0 \times 1 = 0$

(14) $\lim_{x \to 0} \frac{1-\cos x + \sin x}{1 - \cos x - \sin x}$

dividing both numerator and denominator by x

$$= \lim_{x \to 0} \frac{\frac{1-\cos x}{x} + \frac{\sin x}{x}}{\frac{1-\cos x}{x} - \frac{\sin x}{x}} = \frac{0+1}{0-1} = -1$$

(15) $\lim_{x \to 0} \frac{x - x\cos x}{\sin^2 3x}$

dividing both numerator and denominator by x^2

$$= \lim_{x \to 0} \frac{\frac{1-\cos x}{x}}{\frac{\sin^2 3x}{x^2}} = \frac{0}{(3)^2} = 0$$

(16) $\lim_{x \to 0} \frac{\sin x(1-\cos x)}{x^2} = \lim_{x \to 0} \frac{\sin x}{x} \times \frac{1-\cos x}{x}$

$= 1 \times 0 = 0$

(17) $\lim_{x \to 0} \frac{1-\cos^2 3x}{4x^2}$

$= \lim_{x \to 0} \frac{\sin^2 3x}{4x^2} = \lim_{x \to 0} \frac{1}{4} \times \left(\frac{\sin 3x}{x}\right)^2$

$= \frac{1}{4} \times (3)^2 = \frac{9}{4}$

(18) $\lim_{x \to 0} \frac{\sec^2 x - 1}{x^2} = \lim_{x \to 0} \frac{\tan^2 x}{x^2} = (1)^2 = 1$

Remember that:

① $\sin^2 x + \cos^2 x = 1$

② $1 + \tan^2 x = \sec^2 x$

③ $1 + \cot^2 x = \csc^2 x$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٣٢٦٣٦٩٨٢

(19) $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

$= \lim_{x \to 0} \frac{1-\cos x}{x^2} \times \frac{1+\cos x}{1+\cos x}$

$= \lim_{x \to 0} \frac{1-\cos^2 x}{x^2(1+\cos x)} = \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \frac{1}{1+\cos x}$

$= (1)^2 \times \frac{1}{1+1} = 1 \times \frac{1}{2} = \frac{1}{2}$

20) $\lim_{x\to 0} \frac{1-\cos 3x}{\cos^2 2x - 1}$

$= \lim_{x\to 0} \frac{1-\cos 3x}{-\sin^2 2x} \times \frac{1+\cos 3x}{1+\cos 3x}$

$= \lim_{x\to 0} \frac{1}{-\sin^2 2x} \times \frac{1-\cos^2 3x}{1+\cos 3x}$

$= \lim_{x\to 0} \frac{-1}{\sin^2 2x} \times \frac{\sin^2 3x}{1+\cos 3x}$

$= \lim_{x\to 0} \frac{-x^2}{\sin^2 2x} \times \frac{\sin^2 3x}{x^2} \times \frac{1}{1+\cos 3x}$

$= -\left(\frac{1}{2}\right)^2 \times (3)^2 \times \frac{1}{1+1} = -\frac{9}{8}$

21) $\lim_{x\to 0} x(\csc 2x - \cot 3x)$

$= \lim_{x\to 0} x\left(\frac{1}{\sin 2x} - \frac{1}{\tan 3x}\right)$

$= \lim_{x\to 0} \left(\frac{x}{\sin 2x} - \frac{x}{\tan 3x}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

22) $\lim_{x\to 0} 6x^2 \csc 2x \cot x$

$= \lim_{x\to 0} 6 \times \frac{x}{\sin 2x} \times \frac{x}{\tan x} = 6 \times \frac{1}{2} \times 1 = 3$

23 $\lim_{x \to 0} \dfrac{x}{\cos(\frac{\pi}{2} - x)} = \lim_{x \to 0} \dfrac{x}{\sin x} = 1$

(24) $\lim_{x \to \frac{\pi}{2}} \dfrac{2x - \pi}{\cos x}$

$= \lim_{\frac{\pi}{2} - x \to 0} \dfrac{-2(\frac{\pi}{2} - x)}{\sin(\frac{\pi}{2} - x)} = -2$

(25) $\lim_{x \to 1} \dfrac{\sin x\pi}{1 - x}$

$\sin(\pi - x) = \sin x$

$= \lim_{1 - x \to 0} \dfrac{\sin(\pi - \pi x)}{1 - x} = \lim_{1 - x \to 0} \dfrac{\sin \pi(1 - x)}{1 - x} = \pi$

26 $\lim_{x \to 0} \dfrac{x \cot^2 2x}{\csc 3x} = \lim_{x \to 0} \dfrac{x \sin 3x}{\tan^2 2x}$

dividing both denominator and numerator by x^2

$= \lim_{x \to 0} \dfrac{\frac{\sin 3x}{x}}{\frac{\tan^2 2x}{x^2}} = \dfrac{3}{(2)^2} = \dfrac{3}{4}$

(27) $\lim_{(x - \pi) \to 0} \dfrac{\sin x}{x - \pi} = \lim_{x - \pi \to 0} \dfrac{\sin(\pi - x)}{x - \pi}$

$= \lim_{x - \pi \to 0} \dfrac{-\sin(x - \pi)}{(x - \pi)} = -1$

28 $\lim\limits_{x \to -\pi} \dfrac{\tan x}{x+\pi} = \lim\limits_{x+\pi \to 0} \dfrac{\tan(x+\pi)}{x+\pi} = 1$

$(\tan x = \tan(x+\pi))$

29 $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{\pi - 2x}$

$(\tan(\frac{\pi}{2} - x) = \cot x)$

$= \lim\limits_{\frac{\pi}{2} - x \to 0} \dfrac{\tan(\frac{\pi}{2} - x)}{2(\frac{\pi}{2} - x)} = \dfrac{1}{2}$

30 $\lim\limits_{x \to -1} \dfrac{1+x}{\cos \frac{\pi}{2} x}$

$= \lim\limits_{1+x \to 0} \dfrac{1+x}{\sin(\frac{\pi}{2} + \frac{\pi}{2} x)} = \lim\limits_{1+x \to 0} \dfrac{(1+x)}{\sin \frac{\pi}{2}(1+x)} = \dfrac{2}{\pi}$

31 $\lim\limits_{x \to 1} (1-x) \tan \dfrac{\pi x}{2}$

$= \lim\limits_{1-x \to 0} (1-x) \cot(\frac{\pi}{2} - \frac{\pi}{2} x)$

$= \lim\limits_{1-x \to 0} \dfrac{(1-x)}{\tan \frac{\pi}{2}(1-x)} = \dfrac{1}{\frac{\pi}{2}} = \dfrac{2}{\pi}$

32

$$\lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{2x - \pi}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{2x - \pi} \times \frac{1 + \sin x}{1 + \sin x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin^2 x}{(2x - \pi)(1 + \sin x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{(2x - \pi)^2} \times \frac{2x - \pi}{1 + \sin x}$$

$$= \lim_{\frac{\pi}{2} - x \to 0} \frac{\sin^2 (\frac{\pi}{2} - x)}{4 (\frac{\pi}{2} - x)^2} \times \frac{2x - \pi}{1 + \sin x}$$

$$= \frac{1}{4} \times \frac{0}{1 + 0} = \frac{1}{4} \times 0 = \text{zero}$$

33 $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\cos 3x} = \lim_{\frac{\pi}{2} - x \to 0} \frac{\sin(\frac{\pi}{2} - x)}{-\sin(3\frac{\pi}{2} - 3x)}$

$$= \lim_{\frac{\pi}{2} - x \to 0} \frac{-\sin(\frac{\pi}{2} - x)}{(\frac{\pi}{2} - x)} \times \frac{(\frac{\pi}{2} - x)}{\sin 3(\frac{\pi}{2} - x)}$$

$$= -\frac{1}{3}$$

Lesson 6: Existence of the limit of a piecewise defined function at a point

Definition

The function f: $f(x)$ tends to the limit l when $x \to a$ if and only if each of the right and the left limits at a exists and each of them equal l

i.e $\lim_{x \to a} f(x) = l \Leftrightarrow f(a^+) = f(a^-) = l$

Remarks

1. It is not necessary that the function is defined at $x = a$
2. If $f(a^+) \neq f(a^-)$, then $\lim_{x \to a}$ does not exist
3. If the rule of the function is the same on both the right and left of a directly, then the limit of the function can be discussed without discussing the right and left limit

4 If the function f is defined on $]a,b[$ or $[a,b]$ then

* The function is not defined on the left of the point a
  $\Rightarrow \lim_{x \to a^-} f(x)$, and $\lim_{x \to a} f(x)$ donot exist

* The function is not defined on the right of the point b
  $\Rightarrow \lim_{x \to b^+} f(x)$ and $\lim_{x \to b} f(x)$ donot exist

* The limit of the function at the terminal point doesnot exist

* the function has a limit at this point (terminal) from one side only [left or right]

Example 1

If $f(x) = \begin{cases} x-1 & , x \leq 3 \\ 3x-7 & , x > 3 \end{cases}$, find:

(1) $\lim_{x \to 3^-} f(x)$ (2) $\lim_{x \to 3^+} f(x)$

(3) $\lim_{x \to 3} f(x)$

Solution

$$f(3^-) = \lim_{x \to 3^-} (x-1) = 2$$

$$f(3^+) = \lim_{x \to 3^+} (3x-7) = 2$$

$$\because f(3^-) = f(3^+) = 2$$

$$\therefore \lim_{x \to 3} f(x) = 2$$

2 If $f(x) = \begin{cases} x^2+1 & , x < 3 \\ 3x+1 & , x \geq 3 \end{cases}$ find $\lim_{x \to 3} f(x)$

$$f(3^-) = \lim_{x \to 3^-} (x^2+1) = 10$$

$$f(3^+) = \lim_{x \to 3^+} (3x+1) = 10$$

$$\because f(3^-) = f(3^+) \quad \therefore \lim_{x \to 3} f(x) = 10$$

3 If $f(x) = \begin{cases} \dfrac{x^2 - 7x + 12}{x - 3} &, x > 3 \\ 2x - 7 &, x < 3 \end{cases}$

discuss the existence of : $\lim\limits_{x \to 3} f(x)$

Solution:

$$f(3^-) = \lim_{x \to 3^-} \frac{(x-3)(x-4)}{(x-3)} = \lim_{x \to 3^-} x - 4 = -1$$

$$f(3^+) = \lim_{x \to 3^+} (2x - 7) = -1$$

$$\because f(3^-) = f(3^+) = -1 \quad \therefore \lim_{x \to 3} f(x) = -1$$

4 If $f(x) = \begin{cases} \dfrac{3x}{\sin x} &, x < 0 \\ \cos 3x &, x > 0 \end{cases}$

discuss the existence of : $\lim\limits_{x \to 0} f(x)$

Solution

$$f(0^-) = \lim_{x \to 0} \frac{3x}{\sin x} = 3 \lim_{x \to 0} \frac{1}{\frac{\sin x}{x}} = 3$$

$$f(0^+) = \lim_{x \to 0^+} \cos 3x = 1$$

$$\because f(0^-) \neq f(0^+)$$

$$\therefore \lim_{x \to 0} f(x) \text{ doesnot exist}$$

5

If $f(x) = \begin{cases} x-2 & , x < 0 \\ x^2 & , 0 \le x \le 2 \\ 2x & , x > 2 \end{cases}$

discuss the existence of each of the following

(1) $\lim_{x \to 0} f(x)$

(2) $\lim_{x \to 1} f(x)$ (3) $\lim_{x \to 2} f(x)$

Solution

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٣٦٢٩٩٦٨٢

(1) $\because f(0^-)$ doesnot exist

$\therefore \lim_{x \to 0} f(x)$ doesnot exist

(2) The function has the same rule on the right and on the left of $x = 1$ directly and $f(x) = x^2$

$\lim_{x \to 1} x^2 = 1$

(3) $f(2^-) = \lim_{x \to 2^-} x^2 = 4$

$f(2^+) = \lim_{x \to 2^+} 2x = 4$

$\because f(2^-) = f(2^+) = 4$

$\therefore \lim_{x \to 2} f(x) = 4$

6 If $\lim_{x \to 2} f(x) = 7$ where

$$f(x) = \begin{cases} x^2 + 3m & , x < 2 \\ 5x + k & , x > 2 \end{cases}$$

find the value of : m and k

Solution:

$\because \lim_{x \to 2} f(x) = 7$

$\therefore f(2^-) = 7$, $f(2^+) = 7$

$\Rightarrow \lim_{x \to 2^-} (x^2 + 3m) = 7$ | $\lim_{x \to 2^+} (5x + k) = 7$

$4 + 3m = 7 \Rightarrow \boxed{m = 1}$ | $10 + k = 7$

$\Rightarrow \boxed{k = -3}$

7 If $\lim_{x \to 0} f(x) = 2$ where $f(x) = \begin{cases} \frac{\sin 2x}{x} & , x > 0 \\ a \cos 2x & , x < 0 \end{cases}$

find the value of : a

Solution

$\because \lim_{x \to 0} f(x) = 2 \Rightarrow f(0^-) = f(0^+)$

$\Rightarrow \lim_{x \to 0^-} a \cos 2x = \lim_{x \to 0^+} \frac{\sin 2x}{x}$

$\Rightarrow \boxed{a = 2}$

8

If $f(x) = \begin{cases} \dfrac{\sin x + \tan 2x}{x} & , -\frac{\pi}{2} < x < 0 \\ \dfrac{3x}{\tan x} & , 0 < x < \frac{\pi}{2} \end{cases}$

discuss the existence of each of the following:

(1) $\lim\limits_{x \to -\frac{\pi}{2}} f(x)$ (2) $\lim\limits_{x \to 0} f(x)$ (3) $\lim\limits_{x \to \frac{\pi}{3}} f(x)$

Solution:

[1] $f(-\frac{\pi}{2}^-)$ undefined $\therefore$

$\therefore \lim\limits_{x \to -\frac{\pi}{2}} f(x)$ doesnot exist

[2] $\lim\limits_{x \to 0^-} \dfrac{\sin x + \tan 2x}{x} = \lim\limits_{x \to 0^-} \left(\dfrac{\sin x}{x} + \dfrac{\tan 2x}{x} \right)$

$= 1 + 2 = 3$

$\lim\limits_{x \to 0^+} \dfrac{3x}{\tan x} = 3$

$\because f(0^-) = f(0^+) = 3 \quad \therefore \lim\limits_{x \to 0} f(x) = 3$

[3] The function has the same rule on the right and on the left of $x = \frac{\pi}{3}$ directly

and $f(x) = \dfrac{3x}{\tan x}$

$\Rightarrow \lim\limits_{x \to \frac{\pi}{3}} \dfrac{3x}{\tan x} = \dfrac{3 \times \frac{\pi}{3}}{\tan \frac{\pi}{3}} = \dfrac{\pi}{\sqrt{3}} = \dfrac{\pi\sqrt{3}}{3}$

156

⑨ Discuss the existence of the limit of the following:

$$f : f(x) = \begin{cases} \dfrac{3x}{\tan x} & , \ -\frac{\pi}{3} < x < 0 \\ 3\cos x & , \ 0 < x < \frac{\pi}{3} \end{cases}$$ when:

(1) $x \to -\frac{\pi}{3}$ (2) $x \to \frac{\pi}{3}$ (3) $x \to 0$

Solution:

(1) $f(-\frac{\pi}{3}^-)$ is undefined

$\therefore \lim_{x \to -\frac{\pi}{3}} f(x)$ does not exist

(2) $f(\frac{\pi}{3}^+)$ is undefined

$\therefore \lim_{x \to \frac{\pi}{3}} f(x)$ does not exist

(3)

$f(0^-) = \lim_{x \to 0^-} \dfrac{3x}{\tan x} = 3$

$f(0^+) = \lim_{x \to 0^+} 3\cos x = 3$

$\because f(0^-) = f(0^+) = 3$

$\therefore \lim_{x \to 0} f(x) = 3$

If $f(x) = \begin{cases} |x-3| & , x \neq 3 \\ 2 & , x = 3 \end{cases}$,

discuss the existence of : $\lim\limits_{x \to 3} f(x)$

Solution

$|x-3| = \begin{cases} x-3 & , x > 3 \\ -(x-3) & , x < 3 \end{cases}$, $x \neq 3$

$\Rightarrow f(x) = \begin{cases} x-3 & , x > 3 \\ -x+3 & , x < 3 \\ 2 & , x = 3 \end{cases}$

$f(3^-) = \lim\limits_{x \to 3^-} (-x+3) = \text{zero}$

$f(3^+) = \lim\limits_{x \to 3^+} (x-3) = \text{zero}$

$\because f(3^-) = f(3^+) = \text{zero}$

$\therefore \lim\limits_{x \to 3} f(x) = \text{zero}$

If $f(x) = \dfrac{x^2 + 2\sqrt{x^2}}{x}$, discuss the existence of: $\lim\limits_{x \to 0} f(x)$

Solution:

$$\sqrt{x^2} = |x| = \begin{cases} +x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \dfrac{x^2 + 2x}{x} & , x \geq 0 \\ \dfrac{x^2 - 2x}{x} & , x < 0 \end{cases}$$

$$= \begin{cases} x + 2 & , x \geq 0 \\ x - 2 & , x < 0 \end{cases}$$

$$f(0^-) = \lim_{x \to 0^-} (x - 2) = -2$$

$$f(0^+) = \lim_{x \to 0^+} (x + 2) = 2$$

$\because f(0^-) \neq f(0^+)$

$\therefore \lim\limits_{x \to 0} f(x)$ does not exist

12

If the function f where

$$f(x) = \begin{cases} \frac{(x-1)^2}{|x-1|} & , x < 1 \\ 6x - 3m & , x > 1 \end{cases}$$

has a limit at $x = 1$, find the value of m

Solution:

$$f(x) = \begin{cases} \frac{(x-1)^2}{-(x-1)} & , x < 1 \\ 6x - 3m & , x > 1 \end{cases}$$

$$f(x) = \begin{cases} -(x-1) & , x < 1 \\ 6x - 3m & , x > 1 \end{cases}$$

$\because \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow \lim_{x \to 1^-} -(x-1) = \lim_{x \to 1^+} (6x - 3m)$

$\Rightarrow 0 = 6 - 3m$

$\Rightarrow m = 2$

(13)

If $f(x) = \begin{cases} \dfrac{2\sin x}{\pi - x} & , x < \pi \\ 1 - \cos x & , x > \pi \end{cases}$

Find: $\lim\limits_{x \to \pi} f(x)$

Solution:

$$f(\pi^-) = \lim_{x \to \pi^-} \frac{2\sin x}{\pi - x} = \lim_{\pi - x \to 0} \frac{2\sin(\pi - x)}{(\pi - x)} = 2$$

$$f(\pi^+) = \lim_{x \to \pi^+} (1 - \cos x) = 1 - (-1) = 2$$

$$\because f(\pi^-) = f(\pi^+) = 2$$

$$\therefore \lim_{x \to \pi} f(x) = 2$$

(14)

$f(x) = \begin{cases} \dfrac{(x+2)^6 - 64}{x^2 + 32x} & , x < 0 \\ \dfrac{-12x}{1 - (\sin x + \cos x)^2} & , x > 0 \end{cases}$ when $x = 0$

Solution:

$$f(0^-) = \lim_{x \to 0} \frac{(x+2)^6 - (2)^6}{x(x+32)}$$

$$= \lim_{x+2 \to 0} \frac{(x+2)^6 - (2)^6}{(x+2) - 2} \times \frac{1}{x+32} = 6(2)^5 \times \frac{1}{32} = 6$$

$$f(0^+) = \lim_{x \to 0} \frac{-12x}{1 - (\sin^2 x + \cos^2 x + 2\sin x \cos x)}$$

$$= \lim_{x \to 0} \frac{-12x}{-2\sin x \cos x} = \lim_{x \to 0} \frac{6x}{\sin x} \times \frac{1}{\cos x} = 6$$

Find the value of the right and left limit, then deduce the value of the limit for each of the following function at the given point

(1) $f(x) = \sqrt{3-x}$

Solution:

$3 - x \geq 0 \Rightarrow x \geq 3 \Rightarrow x \in [3, \infty[$

∴ the function $f(x)$ is undefined on the left of $x = 3$

i.e $f(3^-)$ is undefined

$f(3^+) = \lim\limits_{x \to 3^+} \sqrt{3-x} = \text{zero}$

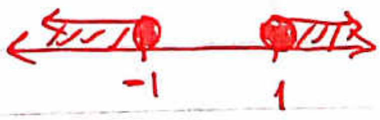
$\lim\limits_{x \to 3} f(x)$ does not exist

(2) $f(x) = \sqrt{x^2-1}$ when $x = -1, x = 1$

Solution:

$x^2 - 1 \geq 0 \Rightarrow x \in \mathbb{R} -]-1, 1[$

$f(-1^-) = \lim\limits_{x \to -1^-} \sqrt{x^2-1} = \text{zero}$



$f(-1^+)$ is undefined $\Rightarrow \lim\limits_{x \to -1} f(x)$ does not exist

$f(1^-)$ is undefined $\Rightarrow f(1^+) = \lim\limits_{x \to 1^+} \sqrt{x^2-1} = \text{zero}$

∴ $\lim\limits_{x \to 1} f(x)$ does not exist

follow 15

③ $f(x) = \sqrt{4-x^2}$ when $x=-2$, $x=2$

Solution:

$$4-x^2 \geq 0 \Rightarrow x^2 \leq 4 \Rightarrow x \in [-2,2]$$

$f(-2^-)$ is undefined

$f(-2^+) = \lim\limits_{x \to -2^+} \sqrt{4-x^2} = \text{zero}$

$\therefore \lim\limits_{x \to -2} f(x)$ doesnot exist

$f(2^-) = \lim\limits_{x \to 2^-} \sqrt{4-x^2} = \text{zero}$

$f(2^+)$ is undefined

$\Rightarrow \lim\limits_{x \to 2} f(x)$ does not exist

$\boxed{16}$ discuss the existence of the limit

$$f(x) = \begin{cases} \dfrac{\sin(\sin x)}{x}, & x > 0 \\ \dfrac{\sqrt[4]{x^2 \tan^2 4x}}{2x}, & x < 0 \end{cases}$$

Solution:

$$f(0^+) = \lim_{x \to 0^+} \frac{\sin(\sin x)}{\sin x} \times \frac{\sin x}{x} = 1$$

$$f(0^-) = \lim_{x \to 0^-} \frac{1}{2} \sqrt[4]{\frac{\tan^2 4x}{x^2}} = \frac{1}{2} \times \sqrt[4]{(4)^2} = \frac{1}{2} \times 2 = 1$$

$$\therefore \lim_{x \to 0} f(x) = 1$$

Lesson 7:

Continuity

Definition:

A function f is said to be continuous at $x = a$ if the, if the following three conditions are sastified together:

(1) The function is defined at $x = a$
i.e. $f(a)$ exists.

(2)
$\lim_{x \to a} f(x)$ exists

(3) $\lim_{x \to a} f(x) = f(a)$

* If the function is defined by more than one rule, then f is continuous at $x = a$

if $f(a^+) = f(a^-) = f(a)$

* Geometrically:

If the curve of the function has open dots or breaks (jumps), then the function is discontinuous function "separated"

Example: Discuss the continuity of each of the following rules at the indicated points:

[1] $f(x) = \frac{x^2-4}{x-2}$ at $x = 1$, $x = 2$

Solution

at $x = 1$

$f(1) = \lim_{x \to 1} \frac{x^2-4}{x-2} = \frac{1-4}{1-2} = 3$

$\therefore$ the function f is continuous at $x = 1$

at $x = 2$

$f(2)$ is undefined

$\therefore$ the function f is not continuous at $x = 2$

[2] $f(x) = 5 - |x-3|$ at $x = 3$

Solution:

$f(3) = \lim_{x \to 3} (5 - |x-3|) = 5$

$\therefore$ the function f is continuous at $x = 3$

[3] $f(x) = \begin{cases} 4x^2+3 & , x \leq 1/2 \\ 5-2x & , x > \frac{1}{2} \end{cases}$

Solution: $f(\frac{1}{2}) = 4(\frac{1}{2})^2 + 3 = 4$ ... (1)

$f(\frac{1}{2}^-) = \lim_{x \to \frac{1}{2}^-} (4x^2+3) = 4$ ... (2)

$f(\frac{1}{2}^+) = \lim_{x \to 1/2^+} (5-2x) = 4$ ... (3)

from 1, 2, 3 f is continuous at $x = 1/2$

(4)

$$f(x) = \begin{cases} x+4 & , x < -2 \\ 1 & , -2 \le x < -1 \\ 2x+3 & , x \geq -1 \end{cases}$$

at $x = -2$, $x = -1$

Solution

$\boxed{\text{at } x = -2}$

$f(-2) = 1$, $f(-2^{+}) = \lim_{x \to -2^{+}} (1) = 1$

$f(-2^{-}) = \lim_{x \to -2^{-}} (x+4) = 2$

$\because f(-2^{-}) \neq f(-2^{+})$

$\therefore$ the function is not continuous at $x = -2$

$\boxed{\text{at } x = -1}$

$f(-1) = 2(-1) + 3 = 1$

$f(-1^{-}) = \lim_{x \to -1} (1) = 1$

$f(-1^{+}) = \lim_{x \to -1^{+}} (2x+3) = 1$

$\because f(-1) = f(-1^{-}) = f(-1^{+})$

the function is continuous at $x = -1$

5

$$f(x)=\begin{cases} x^2+3 & , x\geq 1 \\ \dfrac{x^2+2x-3}{x-1} & , x<1 \end{cases} \quad \text{at } x=1$$

Solution

$$f(1) = (1)^2+3 = 4$$

$$f(1^-) = \lim_{x\to 1^-} \frac{x^2+2x-3}{x-1} = \lim_{x\to 1^-} \frac{(x-1)(x+3)}{(x-1)} = 4$$

$$f(1^+) = \lim_{x\to 1^+} (x^2+3) = 4$$

$$\therefore f(1) = \lim_{x\to 1} f(x)$$

$\therefore f$ is continuous at $x=1$

6

$$f(x)=\begin{cases} x|x| & , x\neq 0 \\ 2 & , x=0 \end{cases} \quad \text{at } x=0$$

Solution:

$$f(x)=\begin{cases} x^2 & , x>0 \\ -x^2 & , x<0 \\ 2 & , x=0 \end{cases}$$

أ/ أحمد عمر
معلم أول رياضيات
٠١٠٢٣٦٣٦٦٨٢

$$f(0) = 2$$

$$f(0^-) = \lim_{x\to 0^-} (-x^2) = 0$$

$$f(0^+) = \lim_{x\to 0^+} (x^2) = 0$$

$$\therefore f(0) \neq \lim_{x\to 0} f(x)$$

$\therefore$ the function is not Continuous

7

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x} & , x > 0 \\ 2\sin x & , x \leq 0 \end{cases} \quad \text{at } x = 0$$

Solution:

$f(0) = 2\sin 0 = 0$

$f(0^-) = \lim\limits_{x \to 0^-} 2\sin x = 0$

$f(0^+) = \lim\limits_{x \to 0^+} \dfrac{1-\cos x}{x} = 0$

$\therefore f(0) = \lim\limits_{x \to 0} f(x)$

$\therefore$ the function is continuous.

8

$$f(x) = \begin{cases} \dfrac{\sqrt{2x+3}-3}{x-3} & , x \neq 3 \\ \frac{1}{3} & , x = 3 \end{cases} \quad \text{at } x = 3$$

$f(3) = \frac{1}{3}$

$$\lim_{x \to 3} \frac{\sqrt{2x+3}-3}{x-3} \times \frac{\sqrt{2x+3}+3}{\sqrt{2x+3}+3}$$

$$= \lim_{x \to 3} \frac{2x+3-9}{(x-3)(\sqrt{2x+3}+3)} = \lim_{x \to 3} \frac{2\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{2x+3}+3)}$$

$$= \frac{2}{6} = \frac{1}{3}$$

$\therefore f(3) = \lim\limits_{x \to 3} f(x)$

$\therefore f$ is continuous at $x = 3$

Example: Find the values of k, a, b and c (168)
so that each of the functions defined by the following rules is continuous at the indicated point.

(1) $f(x) = \begin{cases} \frac{x^2+2x-3}{x+3} & , x \neq -3 \\ x+a & , x = -3 \end{cases}$

$f(-3) = -3+a$

$\lim_{x \to -3} \frac{x^2+2x-3}{x+3} = \lim_{x \to -3} \frac{(x-1)\cancel{(x+3)}}{\cancel{(x+3)}} = -4$

$\because$ f is continuous at $x=-3$ $\therefore f(-3) = \lim_{x \to -3} f(x)$

$\Rightarrow -3+a = -4 \Rightarrow \boxed{a=-1}$

(2) $f(x) = \begin{cases} \frac{x^2-5x+6}{x^3-8} & , x \neq 2 \\ \frac{-2}{|a|} & , x = 2 \end{cases}$ at $x=2$

$\lim_{x \to 2} \frac{x^2-5x+6}{x^3-8} = \lim_{x \to 2} \frac{(x-2)(x-3)}{(x-2)(x^2+2x+4)} = \frac{-1}{12}$

$f(2) = \frac{-2}{|a|}$

$\therefore \frac{-2}{|a|} = \frac{-1}{12} \Rightarrow |a| = 24$

$\Rightarrow a = \pm 24$

③ $f(x) = \begin{cases} \dfrac{\sqrt{x+3}-2}{x^2-1} & , x \neq 1 \\ k & , x = 1 \end{cases}$ at $x = 1$

Solution

$f(1) = k$

$$\lim_{x \to 1} \frac{\sqrt{x+3}-2}{x^2-1} = \lim_{x \to 1} \frac{(x+3)^{\frac{1}{2}} - 4^{\frac{1}{2}}}{(x-1)(x+1)}$$

$$= \lim_{x+3 \to 4} \frac{(x+3)^{\frac{1}{2}} - 4^{\frac{1}{2}}}{[(x+3)-4]} \times \frac{1}{x+1} = \frac{1}{2}(4)^{-\frac{1}{2}} \times \frac{1}{2} = \frac{1}{8}$$

$\because f$ is continuous at $x = 1 \Rightarrow k = \frac{1}{8}$

---

④ $f(x) = \begin{cases} 2 - x^2 & , x \leq c \\ x & , x > c \end{cases}$ at $x = c$

Solution

$\because f$ is continuous at $x = c$

$\therefore f(c^-) = f(c^+)$

$\therefore 2 - c^2 = c \Rightarrow c^2 + c - 2 = 0$

$(c+2)(c-1) = 0$

$\Rightarrow c = -2$ or $c = 1$

Example: Redefine (if possible) each of the functions defined by the following rules at the indicated point, such that each function becomes continuous at this point

[1] $f(x) = \dfrac{x^2 - x - 6}{x - 3}$ at $x = 3$

Solution:

$$\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x \to 3} \frac{(x+2)(x-3)}{x-3} = 5$$

Redefine : $f(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3} & , x \neq 3 \\ 5 & , x = 3 \end{cases}$

$f(x) = \begin{cases} x^3 + 2x & , x > 1 \\ 5x - 1 & , x < 1 \end{cases}$ at $x = 1$

Solution:

$f(1^-) = \lim_{x \to 1^-} (5x - 1) = 4$

$f(1^+) = \lim_{x \to 1^+} (x^3 + 2x) = 3$

$\because f(1^-) \neq f(1^+)$

$\therefore$ the function cannot redfine to be continuous at $x = 1$

Second: Continuity of a function on an interval

Definition:

(1) If the function f is defined on the open interval $I =]a, b[$, then f is continuous on I if it is continuous at each point belongs to this interval.

(2) If the function f is defined on the closed interval $I = [a, b]$, then the function f is continuous on I if

* f is continuous on $]a, b[$
* f is continuous on the right of a

$$f(a) = \lim_{x \to a^+} f(x)$$

* f is continuous from the left at b

$$f(b) = \lim_{x \to b^-} f(x)$$

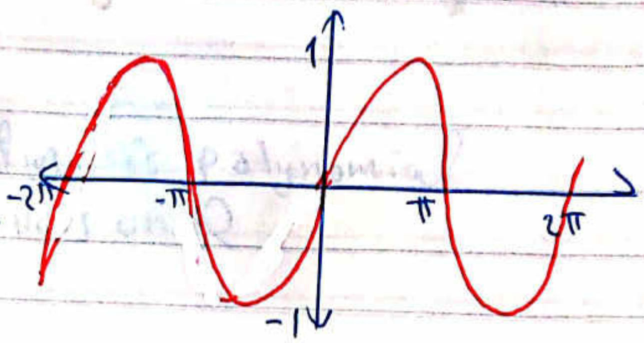
* Some types of continuous functions:

[1] The polynomial functions are continuous on $\mathbb{R}$ or any interval subset of $\mathbb{R}$

[2] The rational function: is continuous on $\mathbb{R} - \{\text{zeroes of denominator}\}$ or in any subset of $\mathbb{R}$ except zeroes of the denominator.

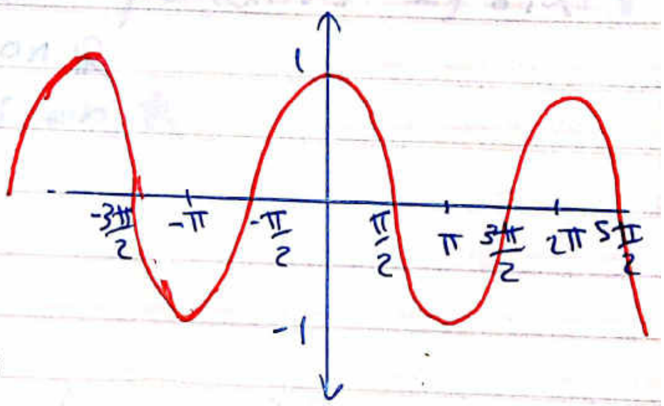
③ The sine function:

$f(x) = \sin x$
is Continuous on $\mathbb{R}$
or any interval subset of $\mathbb{R}$



④ The Cosine function:

$f(x) = \cos x$
is Continuous on $\mathbb{R}$
or any interval subset of $\mathbb{R}$



⑤ The tangent function:

$f(x) = \tan x = \dfrac{\sin x}{\cos x}$

The zeroes of denominator $(\cos x)$ is

$\ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

i.e

the function is continuous on

$\mathbb{R} - \{x : x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}$

Example: Discuss the Continuity of the functions defined by the following rules:

(1) $f(x) = 7$

Solution: $\because$ the function is polynomial

$\therefore$ f is Continuous on $\mathbb{R}$

(2) $f(x) = x^3 \sin 2x$

Solution:

$\because$ each the two functions x^3, $\sin 2x$ are Continuous on $\mathbb{R}$

$\therefore$ f is Continuous on $\mathbb{R}$

(3) $f(x) = \dfrac{x^2 - 9}{x^2 - 5x + 6}$

Solution

$x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$

$\Rightarrow x = 2$ or $x = 3$

$\therefore$ f is Continuous on $\mathbb{R} - \{2, 3\}$

(4) $f(x) = \tan^2 x - 1$

Solution

$\because$ $\tan x$ Continuous on $\mathbb{R} - \{x : x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}$

$\therefore$ f is Continuous on $\mathbb{R} - \{x : x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}$

(5) $f(x) = \dfrac{x}{|x| - 2}$

$\because$ $|x| - 2 = 0 \Rightarrow |x| = 2 \Rightarrow x = \pm 2$

f is Continuous on $\mathbb{R} - \{-2, 2\}$

(6) $f(x) = \dfrac{1}{\sqrt{x-2}}$

Solution:

$\because x - 2 > 0 \Rightarrow x > 2$

$\therefore f$ is continuous on $]2, \infty[$

(7) $f(x) = \dfrac{x^3+1}{\sin x}$

Solution:

f is continuous on $\mathbb{R} - \{$zeroes of denominator$\}$

$\therefore f$ is continuous on $\mathbb{R} - \{x : x = n\pi, n \in \mathbb{Z}\}$

(8) $f(x) = \dfrac{\tan x}{x^2 - 9}$

Solution

$\tan x$ is continuous on $\mathbb{R} - \{x : x = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}$

$, x^2 - 9 = 0 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

$\therefore f$ is continuous on

$\mathbb{R} - \{-3, 3, \frac{\pi}{2} + n\pi\}$ where $n \in \mathbb{Z}$

② $f(x) = \begin{cases} x^2+1 & , x \le 1 \\ 2x & , x > 1 \end{cases}$

Solution:

First we discuss the Continuity on $]-\infty, 1[$ and $]1, \infty[$

$\Rightarrow$ for $x \in]-\infty, 1[$ $f(x) = x^2+1$ is polynomial

$\therefore$ f is continuous on $]-\infty, 1[$

$\Rightarrow$ for $x \in]1, \infty[$, $f(x) = 2x$ is polynomial

$\therefore$ f is continuous on $]1, \infty[$

Second we discuss the continuity at $x = 1$

$f(1) = (1)^2 + 1 = 2$

$f(1^-) = \lim_{x \to 1^-} x^2+1 = 2$

$f(1^+) = \lim_{x \to 1^+} 2x = 2$

$\because$ $f(1) = f(1^-) = f(1^+) = 2$

$\therefore$ f is continuous at $x = 1$

$\therefore$ f is continuous on $\mathbb{R}$

10

$$f(x) = \begin{cases} 1+\sin x & , 0 \le x \le \frac{\pi}{2} \\ 1-\cos 2x & , x > \frac{\pi}{2} \end{cases}$$

Solution:

first we discuss the continuity on $]0, \frac{\pi}{2}[$ and $]\frac{\pi}{2}, \infty[$

$\Rightarrow$ for each $x \in]0, \frac{\pi}{2}[$

$f(x) = 1+\sin x$ is continuous

$\Rightarrow$ for $x \in]\frac{\pi}{2}, \infty[$

$f(x) = 1-\cos 2x$ is continuous

Second we discuss the continuity at $x = \frac{\pi}{2}$

$f(\frac{\pi}{2}) = 1+\sin\frac{\pi}{2} = 2$

$f(\frac{\pi}{2}^-) = \lim\limits_{x \to \frac{\pi}{2}^-} (1+\sin x) = 2$

$f(\frac{\pi}{2}^+) = \lim\limits_{x \to \frac{\pi}{2}^+} (1-\cos 2x) = 2$

$\because f(\frac{\pi}{2}) = f(\frac{\pi}{2}^-) = f(\frac{\pi}{2}^+) = 2$

f is continuous at $x = \frac{\pi}{2}$

third we discuss the continuity at $x=0$ from right

$f(0) = 1 + \sin 0 = 1$

$f(0^+) = \lim\limits_{x \to 0^+} (1 + \sin x) = 1$

$\because f(0) = f(0^+)$

$\therefore f$ is continuous from right at $x = 0$

from first, second and third

f is continuous on $[0, \infty[$

$\boxed{11}$

$$f(x) = \begin{cases} \sqrt{4 - x^2} & , |x| \leq 2 \\ x^2 + 2x & , |x| > 2 \end{cases}$$

Solution:

$$f(x) = \begin{cases} x^2 + 2x & , x < -2 \\ \sqrt{4 - x^2} & , -2 \leq x \leq 2 \\ x^2 + 2x & , x > 2 \end{cases}$$

first we discuss the continuity on $]-\infty, -2[$, $]-2, 2[$ and $]2, \infty[$

$f_1(x) = x^2 + 2x$ is continuous on $]-\infty, -2[$

$f_2(x) = \sqrt{4 - x^2}$ is continuous on $]-2, 2[$

$f_3(x) = x^2 + 2x$ is continuous on $]2, \infty[$

Second) we discuss the continuity at $x=-2$

$f(-2)=\sqrt{4-(-2)^2}=0$

$f(-2^-)=\lim\limits_{x\to -2}(x^2+2x)=0$

$f(-2^+)=\lim\limits_{x\to -2^+}(\sqrt{4-x^2}=0$

$\because f(-2)=f(-2^-)=f(-2^+)=0$

$\therefore f$ is continuous at $x=-2$

third) we discuss the continuity at $x=2$

$f(2)=\sqrt{4-2^2}=0$

$f(2^-)=\lim\limits_{x\to 2^-}\sqrt{4-x^2}=0$

$f(2^+)=\lim\limits_{x\to 2^+}(x^2+2x)=8$

$\because f(2^-)\neq f(2^+)\ \therefore f$ is not continuous at $x=2$

from first, second and third

$f(x)$ is continuous on $R-\{2\}$

(12)

$$f(x) = \begin{cases} \sin x & , -\frac{\pi}{4} \le x < \frac{3\pi}{4} \\ \cos x & , \frac{3\pi}{4} \le x \le 2\pi \end{cases}$$

Solution:

first

$\sin x$ is continuous on $]-\frac{\pi}{4}, \frac{3\pi}{4}[$

$\cos x$ is continuous on $]\frac{3\pi}{4}, 2\pi[$

Second

at $x = \frac{3\pi}{4}$

$f(\frac{3\pi}{4}) = \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$, $f(\frac{3\pi}{4}^+) = \lim\limits_{x \to \frac{3\pi}{4}^+} \cos x = -\frac{1}{\sqrt{2}}$

$f(\frac{3\pi}{3}^-) = \lim\limits_{x \to \frac{3\pi}{3}^-} \sin x = \frac{1}{\sqrt{2}}$

$\because f(\frac{3\pi}{4}^-) \ne f(\frac{3\pi}{4}^+)$ $\therefore$

$\therefore f$ is not continuous at $x = \frac{3\pi}{4}$

third at $x = -\frac{\pi}{4}$ from right

$f(-\frac{\pi}{4}) = \sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

$f(-\frac{\pi}{4}^+) = \lim\limits_{x \to -\frac{\pi}{4}^+} \sin x = -\frac{1}{\sqrt{2}}$

$\because f(-\frac{\pi}{4}) = f(-\frac{\pi}{4}^+)$ $\therefore f$ is continuous from right at $x = -\frac{\pi}{4}$

fourth at $x = 2\pi$ from left

$f(2\pi) = \cos 2\pi = 1$

$f(2\pi^-) = \lim_{x \to 2\pi^-} \cos x = 1$

$\because f(2\pi) = f(2\pi^-)$

$\therefore f$ is continuous from left at $x = 2\pi$

from first, second, third and fourth

f is continuous on $\left[-\frac{\pi}{4}, 2\pi\right] - \left\{\frac{3\pi}{4}\right\}$